4.150

$$\nabla g_1 = \begin{bmatrix} -\frac{P_u}{\sqrt{2}x_1^{\ 2}} - \frac{P_v}{\sqrt{2}\left(x_1+\sqrt{2}x_2\right)^2} \\ -\frac{P_v}{\left(x_1+\sqrt{2}x_2\right)^2} \end{bmatrix}, \quad \boldsymbol{H}g_1 = \begin{bmatrix} \frac{\sqrt{2}P_u}{x_1^{\ 3}} + \frac{\sqrt{2}P_v}{\left(x_1+\sqrt{2}x_2\right)^3} & \frac{2P_v}{\left(x_1+\sqrt{2}x_2\right)^3} \\ \frac{2P_v}{\left(x_1+\sqrt{2}x_2\right)^3} & \frac{2\sqrt{2}P_v}{\left(x_1+\sqrt{2}x_2\right)^3} \end{bmatrix}$$

$$\nabla g_2 = \begin{bmatrix} -\frac{\sqrt{2}P_v}{\left(x_1+\sqrt{2}x_2\right)^2} \\ -\frac{2P_v}{\left(x_1+\sqrt{2}x_2\right)^2} \end{bmatrix}, \quad \boldsymbol{H}g_2 = \begin{bmatrix} \frac{2\sqrt{2}P_v}{\left(x_1+\sqrt{2}x_2\right)^3} & \frac{4P_v}{\left(x_1+\sqrt{2}x_2\right)^3} \\ \frac{4P_v}{\left(x_1+\sqrt{2}x_2\right)^3} & \frac{4\sqrt{2}P_v}{\left(x_1+\sqrt{2}x_2\right)^3} \end{bmatrix}$$

$$\left.\begin{array}{l} \left.\begin{array}{l} g_1 : \text{positive demidefinite} \rightarrow \text{convex} \\ g_2 : \text{positive demidefinite} \rightarrow \text{convex} \\ g_3, g_4 : \text{linear} \rightarrow \text{convex} \end{array}\right\} \rightarrow \text{constraint set is convex} \\ f : \text{linear} \rightarrow \text{convex} \end{array}\right\} \rightarrow \text{convex problem}$$

4.152

$$L = 2\sqrt{2}x_1 + x_2 + u_1\left\{\frac{1}{\sqrt{2}}\left[\frac{P_u}{x_1} + \frac{P_v}{x_1+\sqrt{2}x_2}\right] - 20000\right\} + u_2\left\{\frac{\sqrt{2}P_v}{x_1+\sqrt{2}x_2} - 20000\right\} + u_3\left(-x_1\right) + u_4\left(-x_2\right)$$

$$g_1 = g_2 = 0 (= u_3 = u_4)$$

$$\left.\begin{array}{l} \frac{\partial L}{\partial x_1} = 2\sqrt{2} + u_1\left[-\frac{P_u}{\sqrt{2}x_1^{\ 2}} - \frac{P_v}{\sqrt{2}\left(x_1+\sqrt{2}x_2\right)^2}\right] + u_2\left[-\frac{\sqrt{2}P_v}{\left(x_1+\sqrt{2}x_2\right)^2}\right] = 0 \\ \frac{\partial L}{\partial x_2} = 1 + u_1\left[-\frac{P_v}{\left(x_1+\sqrt{2}x_2\right)^2}\right] + u_2\left[-\frac{2P_v}{\left(x_1+\sqrt{2}x_2\right)^2}\right] = 0 \end{array}\right\} \begin{array}{l} \rightarrow u_2 \geq 0 \rightarrow P_v - 3P_u \geq 0 \\ \rightarrow \tan\theta \geq 3 \end{array}$$

$$\left.\begin{array}{l} g_1 = \frac{1}{\sqrt{2}}\left[\frac{P_u}{x_1} + \frac{P_v}{x_1+\sqrt{2}x_2}\right] - 20000 = 0 \rightarrow x_1 = \frac{P_u}{10000\sqrt{2}} \\ g_2 = \frac{\sqrt{2}P_v}{x_1+\sqrt{2}x_2} - 20000 = 0 \rightarrow x_1 + \sqrt{2}x_2 = \frac{\sqrt{2}P_v}{20000} \end{array}\right\} \rightarrow x_2 = \frac{P_v - P_u}{20000} \geq 0 \rightarrow \tan\theta \geq 1$$

optimal solution only when $\theta \geq 71.57°$

5.50

(0) problem formulation

$$\textit{Minimize} \quad f = 400\left(0.5\pi D^2 + \pi DH\right)$$

$$\textit{subject to} \;\; h_1 = \frac{\pi D^2 H}{4} - 250\pi = 0$$

$$g_1 = H - 8D \le 0$$

(1) check for convexity

$$\nabla f = \begin{bmatrix} 400\pi D + 400\pi H \\ 400\pi D \end{bmatrix}, \quad H = \begin{bmatrix} 400\pi & 400\pi \\ 400\pi & 0 \end{bmatrix}$$

since Hessian of the cbjective function is NOT positive definite, this is NOT a convex programming problem.

(2) KT necessary conditions

$$L = 400\left(0.5\pi D^2 + \pi DH\right) + v_1\left(\frac{\pi D^2 H}{4} - 250\pi\right) + u_1\left(H - 8D\right)$$

$$\frac{\partial L}{\partial D} = 400\pi D + \pi H + v_1 \frac{\pi DH}{2} + u_1(-8) = 0$$

$$\frac{\partial L}{\partial H} = 400\pi D + v_1 \frac{\pi D^2}{4} + u_1 = 0$$

$$h_1 = 0$$

$$g_1 \le 0, \; u_1 g_1 = 0, \; u_1 \ge 0$$

(3) solve the KT conditions

$$i)\, g_1 = 0 \rightarrow H = 8D \rightarrow D = 5, H = 40, v_1 = -226.7, u_1 = -1832.5 < 0(\times)$$

$$ii)\, u_1 = 0 \rightarrow D = H = 10, v_1 = -160, g_1 = H - 8D = -70 < 0 \rightarrow f^* = 60000\pi$$

sufficiency check:

$$\nabla^2 L = \nabla^2 f + v_1 \nabla^2 h_1 = \begin{bmatrix} 400\pi & 400\pi \\ 400\pi & 0 \end{bmatrix} - 160 \begin{bmatrix} 0.5\pi H & 0.5\pi D \\ 0.5\pi D & 0 \end{bmatrix} = \begin{bmatrix} 400\pi - 80\pi H & 400\pi - 80\pi D \\ 400\pi - 80\pi D & 0 \end{bmatrix}$$

$$\nabla^2 L(\boldsymbol{x}^*) = \begin{bmatrix} -400\pi & -400\pi \\ -400\pi & 0 \end{bmatrix} : \text{NOT positive definite!}$$

$$\nabla h_1(\boldsymbol{x}^*)^T \boldsymbol{d} = 25\pi \begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = 0 \rightarrow \boldsymbol{d} = c\begin{bmatrix} 1 & -2 \end{bmatrix}^T$$

$$\boldsymbol{d}^T \nabla^2 L(\boldsymbol{x}^*) \boldsymbol{d} = -400\pi c^2 \begin{bmatrix} 1 & -2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} = 1200\pi c^2 > 0 \rightarrow \text{isolated local minimum}$$

(4) post-optimality analysis

$h_1 : 250\pi \rightarrow 255\pi$

$\Delta f = -v_1 \Delta b = -(-160)(255\pi - 250\pi) = 800\pi$

4.132

1	2	3	4	5	6	7	8	9	10
T	T	T	F	T	F	F	F	F	T