Convolution (1)

$$\left.\begin{aligned} f(x) &= \sum c_k e^{ikx} \\ g(x) &= \sum d_k e^{ikx} \end{aligned}\right\} \rightarrow \text{what are the Fourier coefficients of } f(x)g(x) = \sum h_k e^{ikx}?$$

The answer is not $c_k d_k$. Those are not the coefficients of $h(x) = f(x)g(x)$.

The correct coefficients of $\left(\sum c_k e^{ikx}\right)\left(\sum d_k e^{ikx}\right)$ come from "convolving" the vector of $c's$ with the vector of $d's$.

That convolution is written $\mathbf{c} * \mathbf{d}$.

$$\left.\begin{array}{l} \text{What function does have the Fourier coefficients } c_k d_k? \\ \text{This time we are multiplying in "trasnform space".} \\ \text{In that case we should convolve } f(x) \text{ with } g(x) \text{ in "}x\text{-space"} \end{array}\right\} \xrightarrow{\text{Convolution Rules}} \left\{\begin{array}{c} \text{multiplication} \\ \text{in one space} \\ f(x)g(x) \\ 2\pi c_k d_k \end{array}\right\} \Leftrightarrow \left\{\begin{array}{c} \text{convolution} \\ \text{in the other domain} \\ \mathbf{c} * \mathbf{d} \\ 2\pi c_k d_k \end{array}\right\}$$

The same convolution rules hold for the discrete N-point transforms.

The discrete convolutions have to be "cyclic, so that we write $\mathbf{c} \circledast \mathbf{d}$.

All the sums have N terms, because higher powers of w fold back into lower powers. $\rightarrow w^N = w^0 = 1$

Convolution (2)

$$\left.\begin{aligned} f(w) &= 1+2w+4w^2 \\ g(w) &= 3+5w^2 \end{aligned}\right\} \xrightarrow{\text{convolution}} \begin{cases} \text{non-cyclic: } (1,2,4)*(3,0,5) = (3,6,17,10,20) \\ \text{cyclic: } (1,2,4)*(3,0,5) = (13,26,17) \end{cases}$$

$$c = \underbrace{[1\ 2\ 4]}_{\text{length } L};\ d = \underbrace{[3\ 0\ 5]}_{\text{length } N};\ \underbrace{\text{conv}(c,d)}_{\text{length } L+N-1} \rightarrow (c*d)_n = \sum c_k d_{n-k}$$

$$q = \left[\text{conv}(c,d)\ \ 0\right];\ \text{cconv} = q(1:N) + q(N+1:N+N) \rightarrow (c*d)_n = \sum c_k d_l \ \text{ for } k+l = n (\text{mod } N)$$

$$\text{circulant matrix: } \mathbf{C}_N \mathbf{d} = \begin{bmatrix} c_0 & c_{N-1} & & & c_1 \\ c_1 & c_0 & c_{N-1} & & c_2 \\ c_2 & c_1 & c_0 & & \\ \vdots & & c_1 & c_0 & \\ c_{N-1} & & & c_1 & c_0 \end{bmatrix} \begin{bmatrix} d_0 \\ d_1 \\ \\ \\ d_{N-1} \end{bmatrix} = \mathbf{c} * \mathbf{d}$$

$$\text{Toeplitz matrix: } \mathbf{C}_\infty \mathbf{d} = \begin{bmatrix} & & c_{-2} & & \\ & c_0 & c_{-1} & c_{-2} & \\ c_2 & c_1 & c_0 & c_{-1} & c_{-2} \\ \vdots & c_2 & c_1 & c_0 & \\ & & c_2 & & \end{bmatrix} \begin{bmatrix} d_0 \\ d_1 \\ \\ \\ d_{N-1} \end{bmatrix} = \mathbf{c} * \mathbf{d}$$

Filter (1)

$$x_k \rightarrow \underbrace{\left(\text{averaging filter}\right)}_{\text{smooth the data}} \rightarrow y_k = \frac{x_k + x_{k-1}}{2} \Leftrightarrow \begin{cases} \text{convolution: } y = \underset{\text{filter}}{h} * x \\ y_k = \sum_l h_l x_{k-l} = \cdots + h_0 x_k + h_1 x_{k-1} + \cdots \rightarrow \begin{cases} l = 0 \rightarrow h_0 = \frac{1}{2} \\ l = 1 \rightarrow h_1 = \frac{1}{2} \end{cases} \end{cases}$$

$$\begin{bmatrix} \vdots \\ \\ y_k \\ y_{k+1} \\ \vdots \end{bmatrix} = \begin{bmatrix} \ddots & & & & \\ \ddots & \ddots & & & \\ & 1/2 & 1/2 & & \\ & & 1/2 & 1/2 & \\ & & & \ddots & \ddots \end{bmatrix} \begin{bmatrix} \vdots \\ x_{k-1} \\ x_k \\ x_{k+1} \\ \vdots \end{bmatrix} \xrightarrow{\text{filter}} \begin{cases} \text{LTI}\left(\text{Linear Time Invariant}\right)\text{: keep the filter unchanged} \\ \text{as the signal comes through, keep the half}\left(\text{shift} \rightarrow \text{shift}\right) \\ \text{FIR}\left(\text{Finite Impulse Response}\right)\text{: } x_0 = 1,\ 0 \text{ others} \end{cases}$$

⟨Lowpass Filter⟩

input : $x_k = e^{ik\omega}, -\pi \le \omega \le \pi$

$\omega = 0 :\ x_k = 1 \rightarrow y_k = 1$

$\omega = \pi :\ x_k = \left(e^{i\pi}\right)^k = \left(-1\right)^k = \left(\ldots, 1, -1, 1, -1, \ldots\right) \rightarrow y_k = 0$

in-between frequency : $y_k = \dfrac{e^{ik\omega} + e^{i(k-1)\omega}}{2} = \left(\dfrac{1 + e^{-i\omega}}{2}\right) e^{ik\omega} = \underbrace{H\left(\omega\right)}_{\substack{\text{frequency}\\\text{response}}} e^{ik\omega} \rightarrow \begin{cases} \omega = 0 :\ H\left(\omega\right) = 1 \\ \omega = \pi :\ H\left(\omega\right) = 0 \end{cases}$

Filter (2)

$$x_k \to \underbrace{(\text{differencing filter})}_{\text{smooth the data}} \to y_k = \frac{x_k - x_{k-1}}{2} \Leftrightarrow \begin{cases} \text{convolution: } y = \underset{\text{filter}}{h} * x \\ y_k = \sum_l h_l x_{k-l} = \cdots + h_0 x_k + h_1 x_{k-1} + \cdots \to \begin{cases} l = 0 \to h_0 = \frac{1}{2} \\ l = 1 \to h_1 = -\frac{1}{2} \end{cases} \end{cases}$$

$$\begin{bmatrix} \vdots \\ \\ y_k \\ y_{k+1} \\ \vdots \end{bmatrix} = \begin{bmatrix} \ddots & & & & \\ \ddots & \ddots & & & \\ & -1/2 & 1/2 & & \\ & & -1/2 & 1/2 & \\ & & & \ddots & \ddots \end{bmatrix} \begin{bmatrix} \vdots \\ x_{k-1} \\ x_k \\ x_{k+1} \\ \vdots \end{bmatrix} \xrightarrow{\text{filter}} \begin{cases} \text{LTI (Linear Time Invariant): keep the filter unchanged} \\ \text{as the signal comes through, keep the half (shift} \to \text{shift)} \\ \text{FIR (Finite Impulse Response): } x_0 = 1,\ 0 \text{ others} \end{cases}$$

⟨Highpass Filter⟩

input : $x_k = e^{ik\omega}, -\pi \le \omega \le \pi$

$\omega = 0:\ x_k = 1 \to y_k = 0$

$\omega = \pi:\ x_k = \left(e^{i\pi}\right)^k = (-1)^k = (\ldots,1,-1,1,-1,\ldots) \to y_k = x_k$

$$\text{in-between frequency: } y_k = \frac{e^{ik\omega} - e^{i(k-1)\omega}}{2} = \left(\frac{1 - e^{-i\omega}}{2}\right) e^{ik\omega} = \underbrace{H(\omega)}_{\text{frequency response}} e^{ik\omega} \to \begin{cases} \omega = 0:\ H(\omega) = 0 \\ \omega = \pi:\ H(\omega) = 1 \end{cases}$$

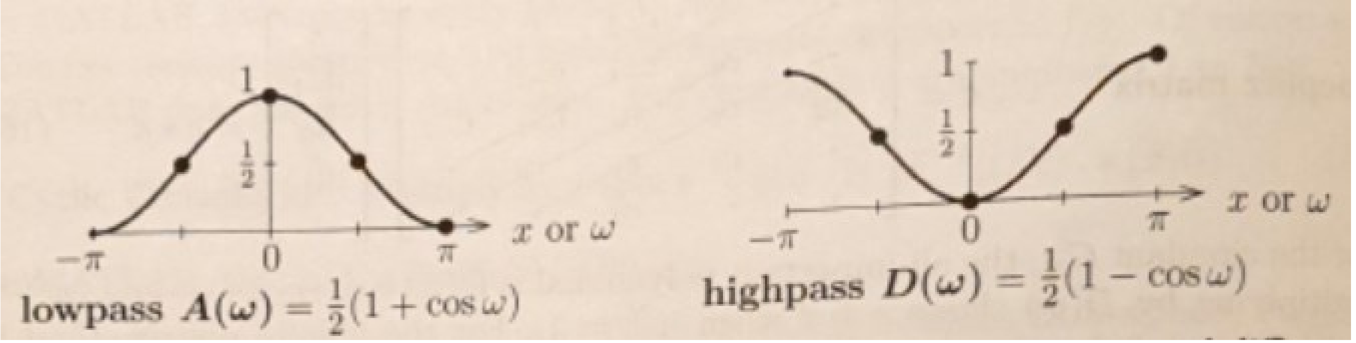
Filter (3)

better filter : averaging twice

$$y_k = \frac{\frac{x_k + x_{k-1}}{2} + \frac{x_{k-1} + x_{k-2}}{2}}{2} = \frac{1}{4} x_k + \frac{1}{2} x_{k-1} + \frac{1}{4} x_{k-2}$$

$$H(\omega) = \frac{1}{4} + \frac{1}{2} e^{-i\omega} + \frac{1}{4} e^{-i2\omega}$$

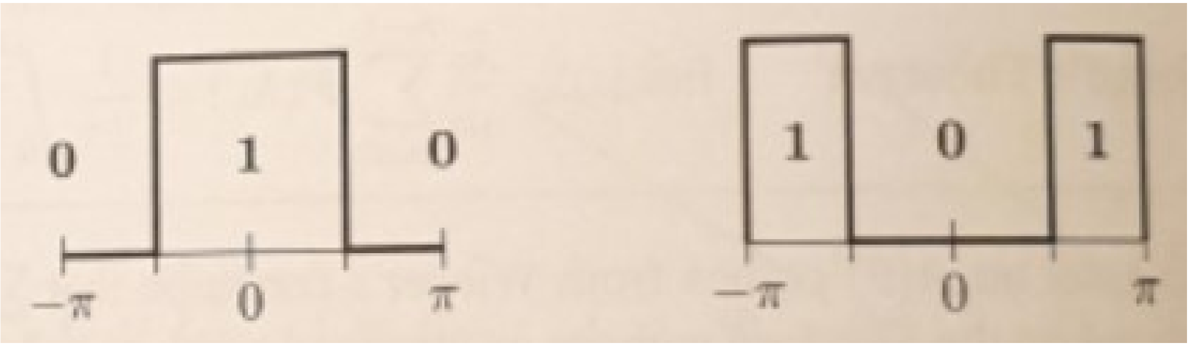
how to visualize convolution?



moving window: picture of $y_k = \sum h_k x_{l-k}$

$x_l \rightarrow h_k x_l \rightarrow \sum \quad \rightarrow$ shift

connection between $\begin{cases} h_k : \text{ numbers in impulse response, filter coeff} \\ H(\omega) : \text{ amplify/diminish in frequency response} \end{cases}$



$$H(\omega) = \sum h_k e^{-ik\omega}$$

$h_k \rightarrow H(\omega)$: Discrete Time Fourier Series

Shift Matrices and Circulant Matrices

upward shift cyclic permutation

$$\mathbf{Px} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_3 \\ x_4 \\ x_1 \end{bmatrix}, \ \mathbf{P}^2\mathbf{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_3 \\ x_4 \\ x_1 \\ x_2 \end{bmatrix}$$

$$\mathbf{P}^4 \left(= \left(\mathbf{P}^3 \right) \mathbf{P} \right) = \mathbf{I} \rightarrow \mathbf{P}^3 = \mathbf{P}^{-1}$$

Circulant matrix: $\mathbf{C} = c_0\mathbf{I} + c_1\mathbf{P} + c_2\mathbf{P}^2 + c_3\mathbf{P}^3 = \begin{bmatrix} c_0 & c_1 & c_2 & c_3 \\ c_3 & c_0 & c_1 & c_2 \\ c_2 & c_3 & c_0 & c_1 \\ c_1 & c_2 & c_3 & c_0 \end{bmatrix}$

Shift Matrices and Circulant Matrices

$\mathbf{CD} = \mathbf{DC}$

$$\mathbf{CD} = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 5 & 0 & 4 \\ 4 & 5 & 0 \\ 0 & 4 & 5 \end{bmatrix} = \begin{bmatrix} 13 & 22 & 19 \\ 19 & 13 & 22 \\ 22 & 19 & 13 \end{bmatrix}$$

When we multiply N by N circulant matrices $\mathbf{C}$ and $\mathbf{D}$, we take the cyclic convolution of the vectors $(c_0, c_1, \ldots, c_{N-1})$ and $(d_0, d_1, \ldots, d_{N-1})$.

Ordinary convolution finds the coefficients when we multiply $(c_0\mathbf{I} + c_1\mathbf{P} + \cdots + c_{N-1}\mathbf{P}^{N-1})$ times $(d_0\mathbf{I} + d_1\mathbf{P} + \cdots + d_{N-1}\mathbf{P}^{N-1})$.

crucial fact: $\mathbf{P}^N = \mathbf{I}$

$$\begin{cases} \text{convolution: } (1,2,3)*(5,0,4) = (5,10,19,8,12) \\ \text{cyclic convolution: } (1,2,3)*(5,0,4) = (13,22,19) \rightarrow \text{quick check?} \end{cases}$$

$$(0,1,0)*(d_0,d_1,d_2) = (d_1,d_2,d_0)$$

$$(1,1,1)*(d_0,d_1,d_2) = (d_0+d_1+d_2, d_0+d_1+d_2, d_0+d_1+d_2)$$

$$(c_0,c_1,c_2)*(d_0,d_1,d_2) = (d_0,d_1,d_2)*(c_0,c_1,c_2)$$

Eigenvalues and Eigenvectors of P

$N = 4,\ \mathbf{Px} = \lambda\mathbf{x}$

$$\mathbf{Px} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_3 \\ x_4 \\ x_1 \end{bmatrix} = \lambda\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \rightarrow \begin{cases} x_2 = \lambda x_1 \\ x_3 = \lambda x_2 \\ x_4 = \lambda x_3 \\ x_1 = \lambda x_4 = \lambda^2 x_3 = \lambda^3 x_2 = \lambda^4 x_1 \rightarrow \lambda^4 = 1 \end{cases}$$

The eigenvalues of $\mathbf{P}$ are the fourth roots of $1 \rightarrow \lambda = i,\ i^2\left(=-1\right),\ i^3\left(=-i\right),\ i^4\left(=1\right)$

The solution to $z^N = 1$ are $\lambda = w, w^2, \ldots, w^{N-1}, 1$ with $w = e^{2\pi i/N}$

eigenvectors for $\lambda = 1,\ i,\ i^2,\ i^3$: $\mathbf{q}_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix},\ \mathbf{q}_1 = \begin{bmatrix} 1 \\ i \\ i^2 \\ i^3 \end{bmatrix},\ \mathbf{q}_2 = \begin{bmatrix} 1 \\ i^2 \\ i^4 \\ i^6 \end{bmatrix},\ \mathbf{q}_3 = \begin{bmatrix} 1 \\ i^3 \\ i^6 \\ i^9 \end{bmatrix}$

$\mathbf{Px} = \lambda\mathbf{x} \rightarrow \lambda\left(\mathbf{P}\right) = 1, w, \ldots w^{N-1},\ \mathbf{q}_k$ is the column of Fourier matrix

Eigenvalues and Eigenvectors of C

$$\mathbf{C}\mathbf{q}_k = \lambda \mathbf{q}_k$$

$$\left(c_0\mathbf{I} + c_1\mathbf{P} + \cdots + c_{N-1}\mathbf{P}^{N-1}\right)\mathbf{q}_k = \left(c_0 + c_1\lambda_k + \cdots + c_{N-1}\lambda_k^{\ N-1}\right)\mathbf{q}_k$$

$\lambda_k = w^k = e^{2\pi ik/N}$: k-th eigenvalue of $\mathbf{P}$

$$\begin{bmatrix} \lambda_0(\mathbf{C}) \\ \lambda_1(\mathbf{C}) \\ \lambda_2(\mathbf{C}) \\ \vdots \\ \lambda_{N-1}(\mathbf{C}) \end{bmatrix} = \begin{bmatrix} c_0 + c_1 + \cdots + c_{N-1} \\ c_0 + c_1 w + \cdots + c_{N-1} w^{N-1} \\ c_0 + c_1 w^2 + \cdots + c_{N-1} w^{2(N-1)} \\ \vdots \\ c_0 + c_1 w^{N-1} + \cdots + c_{N-1} w^{(N-1)(N-1)} \end{bmatrix} = \mathbf{F}\begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_{N-1} \end{bmatrix} = \mathbf{F}\mathbf{c}$$

N eigenvalues of $\mathbf{C}$ are the components of $\mathbf{F}\mathbf{c} =$ inverse Fourier transform of $\mathbf{c}$

$$N = 2, w = e^{2\pi i/2} = -1$$

$$\mathbf{P} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},\ \mathbf{C} = \begin{bmatrix} c_0 & c_1 \\ c_1 & c_0 \end{bmatrix},\ \mathbf{c} = \begin{bmatrix} c_0 \\ c_1 \end{bmatrix},\ \mathbf{F} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\lambda(\mathbf{P}) = \pm 1,\ \lambda\left(\mathbf{C} = c_0\mathbf{I} + c_1\mathbf{P}\right) = c_0 + c_1, c_0 - c_1 \leftrightarrow \mathbf{F}\mathbf{c} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} c_0 + c_1 \\ c_0 - c_1 \end{bmatrix}$$

Convolution Rule

top row of $\mathbf{CD}$ = cyclic convolution = $\mathbf{c} * \mathbf{d} \rightarrow \lambda(\mathbf{CD}) = \mathbf{F}(\mathbf{c} * \mathbf{d})$

$\lambda(\mathbf{C}) = \mathbf{Fc},\ \lambda(\mathbf{D}) = \mathbf{Fd},\ \mathbf{q}_k$ are the same for $\mathbf{C}$ and $\mathbf{D}$

$\lambda(\mathbf{CD}) = \lambda(\mathbf{C})\lambda(\mathbf{D}) = (\mathbf{Fc}).*(\mathbf{Fd})$

$.*$(Hadamard product): component-by-component multiplication

convolution rule: $\mathbf{F}(\mathbf{c} * \mathbf{d}) = (\mathbf{Fc}).*(\mathbf{Fd})$

$$\Lambda(\mathbf{C})\Lambda(\mathbf{D}) = \left(\mathbf{F}^{-1}\mathbf{CF}\right)\left(\mathbf{F}^{-1}\mathbf{DF}\right) = \mathbf{F}^{-1}(\mathbf{CD})\mathbf{F} = \Lambda(\mathbf{CD})$$

Example: $N = 2,\ \mathbf{F} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix},\ \mathbf{F}(\mathbf{c} * \mathbf{d}) = (\mathbf{Fc}).*(\mathbf{Fd})$

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} c_0 d_0 + c_1 d_1 \\ c_0 d_1 + c_1 d_0 \end{bmatrix} = \left(\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} c_0 \\ c_1 \end{bmatrix}\right).*\left(\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} d_0 \\ d_1 \end{bmatrix}\right)$$

Convolution of Functions

periodic function f and g $\left(-\pi \le x \le \pi\right)$, and their Fourier series are infinite $\left(k = 0, \pm 1, \ldots\right)$

what are the Fourier coefficients of $f\left(x\right)g\left(x\right)$?

$$f\left(x\right)g\left(x\right) = \left(\sum_{k=-\infty}^{\infty} c_k e^{ikx}\right)\left(\sum_{m=-\infty}^{\infty} d_m e^{imx}\right) = \sum_{n=-\infty}^{\infty} h_n e^{inx}$$

The coefficients of h_n combines all products of $c_k d_m$ with $k + m = n$

$h_n = \sum_{k=-\infty}^{\infty} c_k d_{n-k}$ is convolution $\mathbf{h} = \mathbf{c} * \mathbf{d}$ for infinite vectors.

convolution of 2π-periodic functions: $\left(f * g\right)\left(x\right) = \int_{t=-\pi}^{\pi} f\left(t\right) g\left(x-t\right) dt$

convolution rule for periodic functions: Fourier coefficients of $f * g$ are $2\pi c_k d_k$

check with $f\left(x\right) = \sin x$

$$\int_0^{2\pi} \left(f * g\right)\left(x\right) e^{-ikx} dx = \int_0^{2\pi}\left[\int_0^{2\pi} f\left(t\right) g\left(x-t\right) e^{-ik\left[t+\left(x-t\right)\right]} dt\right] dx = \underbrace{\int_0^{2\pi} f\left(t\right) e^{-ikt} dt}_{2\pi c_k} \underbrace{\int_0^{2\pi} g\left(x-t\right) e^{-ik\left(x-t\right)} dt}_{2\pi d_k}$$