1.

Minimize the cross-sectional area:

$$\left.\begin{aligned} &Min\ f = 2(0.1w) + 0.1(h-0.2) = 0.2w + 0.1h - 0.02 \\ &subject\ to\ \ g_1 = -w + 0.1 < 0 \\ &\qquad\qquad\quad g_2 = w - 10 < 0 \\ &\qquad\qquad\quad g_3 = -h + 0.2 < 0 \\ &\qquad\qquad\quad g_4 = h - 10 < 0 \end{aligned}\right\} \rightarrow \begin{cases} (w^*, h^*) = (0.1, 0.2) \\ f^* = 0.02 \end{cases}$$

Minimize the normal stresses:

$$\left.\begin{aligned} &Min\ f = \sigma = \frac{My}{I} = \frac{(1)(h/2)}{\dfrac{0.1(h-0.2)^3}{12} + 2\left[\dfrac{w(0.1)^3}{12} + 0.1w\left(\dfrac{h}{2} - 0.05\right)^2\right]} \\ &subject\ to\ \ g_1 = -w + 0.1 < 0 \\ &\qquad\qquad\quad g_2 = w - 10 < 0 \\ &\qquad\qquad\quad g_3 = -h + 0.2 < 0 \\ &\qquad\qquad\quad g_4 = h - 10 < 0 \end{aligned}\right\} \rightarrow \begin{cases} (w^*, h^*) = (10, 10) \\ f^* = 0.088 \end{cases}$$

2. (sensitivity calculation)

$$(1)\ \boldsymbol{K} = 10^6\begin{bmatrix} 3.0 & -0.6 \\ -0.6 & 0.6 \end{bmatrix},\ \boldsymbol{Y} = \begin{Bmatrix} Y_1 \\ Y_2 \end{Bmatrix},\ \boldsymbol{P} = \begin{Bmatrix} 100 \\ 200 \end{Bmatrix} \rightarrow \boldsymbol{KY} = \boldsymbol{P} \rightarrow \boldsymbol{Y} = 10^{-3}\begin{Bmatrix} 0.1250 \\ 0.4583 \end{Bmatrix}$$

$$(2)\ \boldsymbol{K}\frac{\partial \boldsymbol{Y}}{\partial \boldsymbol{A}} = \frac{\partial \boldsymbol{P}}{\partial \boldsymbol{A}} - \frac{\partial \boldsymbol{K}}{\partial \boldsymbol{A}}\boldsymbol{Y} \rightarrow \boldsymbol{K}\frac{\partial \boldsymbol{Y}}{\partial \boldsymbol{A}} = -\frac{\partial \boldsymbol{K}}{\partial \boldsymbol{A}}\boldsymbol{Y} \rightarrow \begin{cases} \dfrac{\partial \boldsymbol{K}}{\partial A_1} = \begin{bmatrix} E_1/l_1 & 0 \\ 0 & 0 \end{bmatrix} \rightarrow \dfrac{\partial \boldsymbol{Y}}{\partial A_1} = 10^{-4}\begin{Bmatrix} -0.6250 \\ -0.6250 \end{Bmatrix} \\ \dfrac{\partial \boldsymbol{K}}{\partial A_2} = \begin{bmatrix} E_2/l_2 & -E_2/l_2 \\ -E_2/l_2 & E_2/l_2 \end{bmatrix} \rightarrow \dfrac{\partial \boldsymbol{Y}}{\partial A_2} = 10^{-3}\begin{Bmatrix} -0.0000 \\ -0.3333 \end{Bmatrix} \end{cases}$$

$$(3)\ \boldsymbol{\sigma} = \begin{bmatrix} E_1/l_1 & 0 \\ -E_2/l_2 & E_2/l_2 \end{bmatrix}\boldsymbol{Y} \rightarrow \frac{\partial \boldsymbol{\sigma}}{\partial \boldsymbol{A}} = \frac{\partial \boldsymbol{\sigma}}{\partial \boldsymbol{Y}}\frac{\partial \boldsymbol{Y}}{\partial \boldsymbol{A}} \rightarrow \begin{cases} \dfrac{\partial \boldsymbol{\sigma}}{\partial A_1} = \begin{Bmatrix} -75 \\ 0 \end{Bmatrix} \\ \dfrac{\partial \boldsymbol{\sigma}}{\partial A_2} = \begin{Bmatrix} 0 \\ -200 \end{Bmatrix} \end{cases}$$

3.

Minimize $P(u)=\int_0^1\sqrt{1+(u')^2}\,dx$ with $u(0)=a, u(1)=b, \int_0^1 u(x)dx=A$

$$L=\int_0^1\sqrt{1+(u')^2}\,dx+\lambda\left(\int_0^1 u(x)dx-A\right)=\int_0^1\left[\sqrt{1+(u')^2}+\lambda u\right]dx-\lambda A$$

$$\frac{\partial F}{\partial u}-\frac{d}{dx}\left(\frac{\partial F}{\partial u'}\right)=0\rightarrow\lambda-\frac{d}{dx}\left[\frac{u'}{\sqrt{1+(u')^2}}\right]=0\rightarrow\lambda x-\frac{u'}{\sqrt{1+(u')^2}}=c$$

$$u'=\frac{\lambda x-c}{\sqrt{1-(\lambda x-c)^2}}\rightarrow u(x)=-\frac{1}{\lambda}\sqrt{1-(\lambda x-c)^2}+d\rightarrow(\lambda x-c)^2+(\lambda u-d)^2=1$$

The shortest path is a circular arc.

Three numbers $\lambda,\ c,\ d$ are determined by the conditions $u(0)=a,\ u(1)=b$ and $\int_0^1 u(x)dx=A$

4.

$$\left.\begin{array}{l}\min\limits_{\boldsymbol{x}}: c(\boldsymbol{x})=\boldsymbol{U}^T\boldsymbol{K}\boldsymbol{U}=\sum_{e=1}^{N}(x_e)^p\, u_e^T k_0 u_e\\ \text{subject to: } \dfrac{V(\boldsymbol{x})}{V_0}=f\\ \qquad\qquad : \boldsymbol{K}\boldsymbol{U}=f\\ \qquad\qquad : 0\le x_{\min}\le x\le 1\end{array}\right\}\rightarrow L=c+\lambda\left(\frac{V(\boldsymbol{x})}{V_0}-f\right)+\boldsymbol{\lambda}_l(x_{\min}-x)+\boldsymbol{\lambda}_u(x-1)$$

$$\begin{cases}\dfrac{\partial L}{\partial x_e}=\dfrac{\partial c}{\partial x_e}+\lambda\dfrac{\partial V}{\partial x_e}-\boldsymbol{\lambda}_l+\boldsymbol{\lambda}_u=0\xrightarrow{\text{for intermediate } x_e}B_e=\dfrac{-\dfrac{\partial c}{\partial x_e}}{\lambda\dfrac{\partial V}{\partial x_e}}=1\ \text{ where }\ \dfrac{\partial c}{\partial x_e}=p(x_e)^{p-1}u_e^T k_0 u_e\\ \lambda\left(\dfrac{V(\boldsymbol{x})}{V_0}-f\right)=0\rightarrow\lambda\ \text{ for }\ \dfrac{V(\boldsymbol{x})}{V_0}=f\\ \boldsymbol{\lambda}_l(x_{\min}-x)=0\xrightarrow{\text{for intermediate } x_e}\boldsymbol{\lambda}_l=0\\ \boldsymbol{\lambda}_u(x-1)=0\xrightarrow{\text{for intermediate } x_e}\boldsymbol{\lambda}_u=0\end{cases}$$

$$\rightarrow x_e^{new}=\begin{cases}\max(x_{\min},x_e-m) & \text{if } x_eB_e^{\eta}\le\max(x_{\min},x_e-m)\\ x_eB_e^{\eta} & \text{if } \max(x_{\min},x_e-m)\le x_eB_e^{\eta}\le\min(1,x_e+m)\\ \min(1,x_e+m) & \text{if } \min(1,x_e+m)\le x_eB_e^{\eta}\end{cases}$$

m: move limit, η: numerical damping coefficient