CHAPTER

2

Optimum Design Problem Formulation

Upon completion of this chapter, you will be able to:

- Translate a descriptive statement of the design problem into a mathematical statement for optimization
- Identify and define the problem's design variables
- Identify and define an optimization criterion for the problem
- Identify and define the design problem's constraints
- Transcribe the problem formulation into a standard model for design optimization

It is generally accepted that the *proper definition and formulation of a problem* take more than 50% of the total effort needed to solve it. Therefore, it is critical to follow well-defined procedures for formulating design optimization problems. In this chapter, we describe the process of transforming the design of a selected system and/or subsystem into an optimum design problem. Methods for solving the problem will be discussed in subsequent chapters; here we focus on properly formulating the problem as an optimization problem.

Several simple and moderately complex applications are discussed in this chapter to illustrate the problem formulation process. More advanced applications are discussed in Chapters 6 and 7 and 14–19.

The *importance of properly formulating* a design optimization problem must be stressed because the optimum solution will be only as good as the formulation. For example, if we forget to include a critical constraint in the formulation, the optimum solution will most likely violate it. Also, if we have too many constraints, or if they are inconsistent, there may be no solution for the problem. However, once the problem is properly formulated, good software is usually available to solve it.

It is important to note that the process of developing a proper formulation for optimum design of practical problems is iterative in itself. Several iterations usually are needed to revise

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the formulation before an acceptable one is finalized. This iterative process is further discussed in chapter: Optimum Design: Numerical Solution Process and Excel Solver.

For most design optimization problems, we will use the following *five-step* procedure to formulate the problem:

Step 1: Project/problem description
Step 2: Data and information collection
Step 3: Definition of design variables
Step 4: Optimization criterion
Step 5: Formulation of constraints

Formulation of an optimum design problem implies translating a descriptive statement of the problem into a well-defined mathematical statement.

2.1 THE PROBLEM FORMULATION PROCESS

We will describe the tasks to be performed in each of the foregoing five steps to develop a mathematical formulation for the design optimization problem. These steps are illustrated with some examples in this section and in later sections.

2.1.1 Step 1: Project/Problem Description

Are the Project Goals Clear?

The formulation process begins by developing a descriptive statement for the project/problem, usually by the project's owner/sponsor. The statement describes the overall *objectives* of the project and the *requirements* to be met. This is also called the *statement of work*.

EXAMPLE 2.1 DESIGN OF A CANTILEVER BEAM, PROBLEM DESCRIPTION

Cantilever beams are used in many practical applications in civil, mechanical, and aerospace engineering. To illustrate the step of problem description, we consider the design of a hollow square-cross-section *cantilever beam* to support a load of 20 kN at its end. The beam, made of steel, is 2 m long, as shown in Fig. 2.1. The failure conditions for the beam are as follows: (1) the material should not fail under the action of the load, and (2) the deflection of the free end should be no more than 1 cm. The width-to-thickness ratio for the beam should be no more than 8 to avoid local buckling of the walls. A *minimum-mass* beam is desired. The width and thickness of the beam must be within the following limits:

$$60 \le \text{width} \le 300 \text{ mm} \quad \text{(a)}$$

$$3 \le \text{thickness} \le 15 \text{ mm} \quad \text{(b)}$$

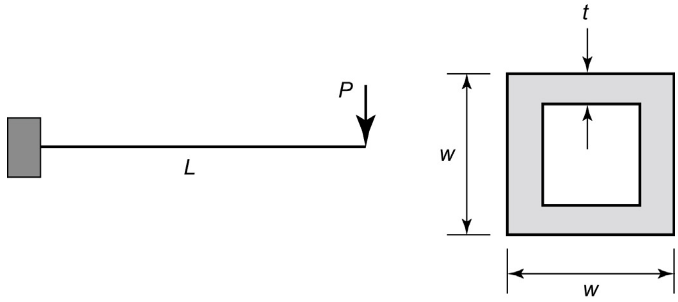

FIGURE 2.1 **Cantilever beam of a hollow square cross-section.**

2.1.2 Step 2: Data and Information Collection

Is all the Information Available to Solve the Problem?

To develop a mathematical formulation for the problem, we need to gather information on material properties, performance requirements, resource limits, cost of raw materials, and so forth. In addition, most problems require the capability to *analyze trial designs*. Therefore, *analysis procedures* and *analysis tools* must be identified at this stage. For example, the finite-element method is commonly used for analysis of structures, so the software tool available for such an analysis needs to be identified. In many cases, the project statement is vague, and assumptions about modeling of the problem need to be made in order to formulate and solve it.

EXAMPLE 2.2 DATA AND INFORMATION COLLECTION FOR CANTILEVER BEAM

The information needed for the *cantilever beam design problem* of Example 2.1 includes expressions for bending and shear stresses, and the expression for the deflection of the free end. The notation and data for this purpose are defined in Table 2.1.

The following are useful expressions for the beam:

$$A = w^2 - (w-2t)^2 = 4t(w-t),\ \text{mm}^2 \tag{c}$$

$$I = \frac{1}{12} w \times w^3 - \frac{1}{12}(w-2t)\times(w-2t)^3 = \frac{1}{12}w^4 - \frac{1}{12}(w-2t)^4,\ \text{mm}^4 \tag{d}$$

$$Q = \frac{1}{2}w^2 \times \frac{w}{4} - \frac{1}{2}(w-2t)^2 \times \frac{(w-2t)}{4} = \frac{1}{8}w^3 - \frac{1}{8}(w-2t)^3,\ \text{mm}^3 \tag{e}$$

$$M = PL,\ \text{N/mm} \tag{f}$$

$$V = P,\ \text{N} \tag{g}$$

TABLE 2.1 Notation and Data for Cantilever Beam

Notation	Data
A	Cross-sectional area, mm^2
E	Modulus of elasticity of steel, 21×10^4 N/mm^2
G	Shear modulus of steel, 8×10^4 N/mm^2
I	Moment of inertia of the cross-section, mm^4
L	Length of the member, 2000 mm
M	Bending moment, N/mm
P	Load at the free end, 20,000 N
Q	Moment about the neutral axis of the area above the neutral axis, mm^3
q	Vertical deflection of the free end, mm
q_a	Allowable vertical deflection of the free end, 10 mm
V	Shear force, N
w	Width (depth) of the section, mm
t	Wall thickness, mm
σ	Bending stress, N/mm^2
σ_a	Allowable bending stress, 165 N/mm^2
τ	Shear stress, N/mm^2
τ_a	Allowable shear stress, 90 N/mm^2

$$\sigma = \frac{Mw}{2I}, \text{ N/mm}^2 \tag{h}$$

$$\tau = \frac{VQ}{2It}, \text{ N/mm}^2 \tag{i}$$

$$q = \frac{PL^3}{3EI}, \text{ mm} \tag{j}$$

2.1.3 Step 3: Definition of Design Variables

What are these Variables?

HOW DO I IDENTIFY THEM?

The next step in the formulation process is to identify a set of variables that describe the system, called the *design variables.* In general, these are referred to as *optimization variables* or simply *variables* that are regarded as *free* because we should be able to assign any value to them. Different values for the variables produce different designs. The design variables should be independent of each other as far as possible. If they are dependent, their values cannot be specified independently because there are constraints between them. The number of independent design variables gives the *design degrees of freedom* for the problem.

For some problems, different sets of variables can be identified to describe the same system. Problem formulation will depend on the selected set. We will present some examples later in this chapter to elaborate on this point.

Once the design variables are given numerical values, we have a *design of the system*. Whether this design *satisfies all requirements* is another question. We will introduce a number of concepts to investigate such questions in later chapters.

If proper design variables are not selected for a problem, the formulation will be either incorrect or not possible. At the initial stage of problem formulation, all options for specification of design variables should be investigated. Sometimes it may be desirable to designate more design variables than apparent design degrees of freedom. This gives added flexibility to problem formulation. Later, it is possible to assign a fixed numerical value to any variable and thus eliminate it from the formulation.

At times it is difficult to clearly identify a problem's design variables. In such a case, a complete list of all variables may be prepared. Then, by considering each variable individually, we can determine whether or not it can be treated as an *optimization variable*. If it is a valid design variable, the designer should be able to specify a numerical value for it to select a trial design.

We will use the term "design variables" to indicate all optimization variables for the optimization problem and will represent them in the vector $\mathbf{x}$. To summarize, the following considerations should be given in identifying design variables for a problem:

- **Generally, the design variables should be independent of each other. If they are not, there must be some equality constraints between them (explained later in several examples).**
- **A minimum number of design variables is required to properly formulate a design optimization problem.**
- **As many independent parameters as possible should be designated as design variables at the problem formulation phase. Later on, some of these variables can be assigned fixed numerical values.**
- **A numerical value should be given to each identified design variable to determine if a trial design of the system is specified.**

EXAMPLE 2.3 DESIGN VARIABLES FOR CANTILEVER BEAM

Only dimensions of the cross-section are identified as design variables for the *cantilever beam design problem* of Example 2.1; all other parameters are specified:

w = outside width (depth) of the section, mm
t = wall thickness, mm

Note that the design variables are defined precisely including the units to be used for them.

It is also noted here that an *alternate set of design variables* can be selected: w_o = outer width of the section, and w_i = inner width of the section. The problem can be formulated using these design variables. However, note that all the expressions given in Eqs. (c)–(j) will have to be re-derived in terms of w_o and w_i. Thus the two formulations will look quite different from each other for the same design problem. However, these two formulations should yield same final solution.

Note also that the wall thickness t can also be specified as a design variable in addition to w_o and w_i. In terms of these variables, the problem formulation will look quite different from the previous two formulations. However, in this case an additional constraint $t = 0.5(w_o - w_i)$ must be imposed in the formulation; otherwise the formulation will not be proper and will not yield a meaningful solution for the problem.

To demonstrate calculation of various analysis quantities, let us select a trial design as w = 60 mm and t = 10 mm and calculate the quantities defined in Eqs. (c)–(j):

$$A = 4t(w-t) = 4(10)(60-10) = 2{,}000\,\text{mm}^2 \tag{k}$$

$$I = \frac{1}{12}w^4 - \frac{1}{12}(w-2t)^4 = \frac{1}{12}(60)^4 - \frac{1}{12}(60-2\times 10)^4 = 866{,}667\,\text{mm}^4 \tag{l}$$

$$Q = \frac{1}{8}w^3 - \frac{1}{8}(w-2t)^3 = \frac{1}{8}(60)^3 - \frac{1}{8}(60-2\times 10)^3 = 19{,}000\,\text{mm}^3 \tag{m}$$

$$M = PL = 20{,}000\times 2{,}000 = 4\times 10^7\ \text{N/mm} \tag{n}$$

$$V = P = 20{,}000\,\text{N} \tag{o}$$

$$\sigma = \frac{Mw}{2I} = \frac{4\times 10^7(60)}{2\times 866{,}667} = 1{,}385\,\text{N/mm}^2 \tag{p}$$

$$\tau = \frac{VQ}{2It} = \frac{20{,}000\times 19{,}000}{2\times 866{,}667\times 10} = 21.93\,\text{N/mm}^2 \tag{q}$$

$$q = \frac{PL^3}{3EI} = \frac{20{,}000\times (2{,}000)^3}{3\times 21\times 10^4\times 866{,}667} = 262.73\,\text{mm} \tag{r}$$

2.1.4 Step 4: Optimization Criterion

How Do I Know that My Design is the Best?

There can be many feasible designs for a system, and some are better than others. The question is how do we quantify this statement and designate a design as better than another. For this, we must have a criterion that associates a number with each design. This way, the merit of a given design is specified. The criterion must be a scalar function whose numerical value can be obtained once a design is specified; that is, it must be a *function of the design variable vector* **x**. Such a criterion is usually called an *objective function* for the optimum design problem, and it needs to be *maximized* or *minimized* depending on problem requirements. A criterion that is to be minimized is usually called a *cost function* in engineering literature, which is the term used throughout this text. It is emphasized that a *valid objective function must be influenced directly or indirectly by the variables of the design problem*; otherwise, it is not a meaningful objective function.

The selection of a proper objective function is an important decision in the design process. Some common objective functions are cost (to be minimized), profit (to be maximized), weight (to be minimized), energy expenditure (to be minimized), and ride quality of a vehicle (to be maximized). In many situations, an obvious objective function can be identified. For example, we always want to minimize the cost of manufacturing goods or maximize return on investment. In some situations, two or more objective functions may be identified. For example, we may want to minimize the weight of a structure and at the same time minimize the deflection or stress at a certain point. These are called *multiobjective design optimization problems* and are discussed in chapter: Multi-objective Optimum Design Concepts and Methods.

For some design problems, it is not obvious what the objective function should be or how it should be expressed in terms of the design variables. Some insight and experience may be needed to identify a proper objective function for a particular design problem. For example, consider the optimization of a passenger car. What are the design variables? What is the objective function, and what is its functional form in terms of the design variables? This is a practical design problem that is quite complex. Usually, such problems are divided into several smaller subproblems and each one is formulated as an optimum design problem. For example, design of a passenger car can be divided into a number of optimization subproblems involving the trunk lid, doors, side panels, roof, seats, suspension system, transmission system, chassis, hood, power plant, bumpers, and so on. Each subproblem is now manageable and can be formulated as an optimum design problem.

EXAMPLE 2.4 OPTIMIZATION CRITERION FOR CANTILEVER BEAM

For the *design problem in Example 2.1*, the objective is to design a minimum-mass cantilever beam. Since the mass is proportional to the cross-sectional area of the beam, the objective function for the problem is taken as the cross-sectional area which is to be minimized:

$$f(w,t) = A = 4t(w-t),\ \text{mm}^2 \tag{s}$$

At the trial design w = 60 mm and t = 10 mm, the cost function is evaluated as

$$f(w,t) = 4t(w-t) = 4\times 10(60-10) = 2{,}000\ \ \text{mm}^2$$

2.1.5 Step 5: Formulation of Constraints

What Restrictions Do I have on My Design?

All restrictions placed on the design are collectively called *constraints*. The final step in the formulation process is to identify all constraints and develop expressions for them. Most realistic systems must be designed and fabricated with the given *resources* and must meet *performance requirements*. For example, structural members should not fail under normal operating loads. The vibration frequencies of a structure must be different from the operating frequency of the machine it supports; otherwise, resonance can occur and cause catastrophic failure. Members must fit into the available space, and so on.

These constraints, as well as others, must depend on the design variables, since only then do their values change with different trial designs; that is, a meaningful constraint must be a function of at least one design variable. Several concepts and terms related to constraints are explained next.

Linear and Nonlinear Constraints

Many constraint functions have only first-order terms in design variables. These are called *linear constraints*. *Linear-programming problems* have only linear constraints and objective functions. More general problems have nonlinear objective function and/or constraint functions. These are called *nonlinear-programming problems*. Methods to treat both linear and nonlinear constraints and objective functions are presented in this text.

Feasible Design

The design of a system is a set of numerical values assigned to the design variables (ie, a particular design variable vector $\mathbf{x}$). Even if this design is absurd (eg, negative radius) or inadequate in terms of its function, it can still be called a design. Clearly, some designs are useful and others are not. A design meeting all requirements is called a *feasible design* (*acceptable* or *workable*). An *infeasible design* (*unacceptable*) does not meet one or more of the requirements.

Equality and Inequality Constraints

Design problems may have equality as well as inequality constraints. The problem description should be studied carefully to determine which requirements need to be formulated as equalities and which ones as inequalities. For example, a machine component may be required to move precisely by Δ to perform the desired operation, so we must treat this as an equality constraint. A feasible design must satisfy precisely all equality constraints. Also, most design problems have inequality constraints, sometimes called *unilateral* or *one-sided constraints*. Note that the *feasible region* with respect to an inequality constraint is much larger than that with respect to the same constraint expressed as equality.

To illustrate the difference between equality and inequality constraints, we consider a constraint written in both equality and inequality forms. Fig. 2.2a shows the equality constraint $x_1 = x_2$. Feasible designs with respect to the constraint must lie on the straight line A–B. However, if the constraint is written as an inequality $x_1 \le x_2$, the feasible region is much larger, as shown in Fig. 2.2b. Any point on the line A–B or above it gives a feasible design. Therefore, it is important to properly identify equality and inequality constraints; otherwise a meaningful solution may not be obtained for the problem.

EXAMPLE 2.5 CONSTRAINTS FOR CANTILEVER BEAM

Using various expressions given in Eqs. (c)–(j), we formulate the constraints for the *cantilever beam design problem* from Example 2.1 as follows:

Bending stress constraint: $\sigma \le \sigma_a$

$$\frac{PLw}{2I} \le \sigma_a \tag{t}$$

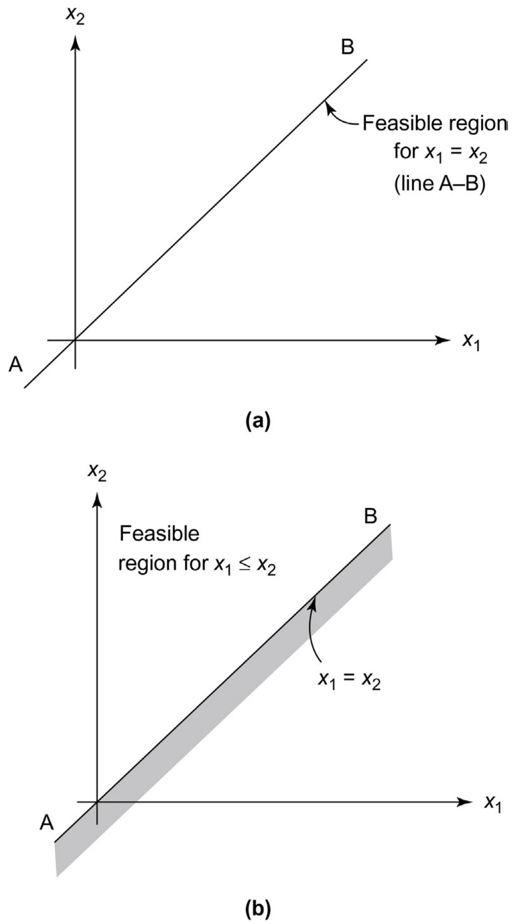

FIGURE 2.2 Shown here is the distinction between equality and inequality constraints. (a) Feasible region for constraint $x_1 = x_2$ (line A − B); (b) feasible region for constraint $x_1 \le x_2$ (line A − B and the region above it).

Shear stress constraint: $\tau \le \tau_a$

$$\frac{PQ}{2It} \le \tau_a \tag{u}$$

Deflection constraint: $q \le q_a$

$$\frac{PL^3}{3EI} \le q_a \tag{v}$$

Width–thickness restriction: $\dfrac{w}{t} \le 8$

$$w \le 8t \tag{w}$$

Dimension restrictions:

$$60 \le w, \text{mm}; \ w \le 300, \text{mm} \tag{x}$$

$$3 \le t, \text{mm}; \ t \le 15, \text{mm} \tag{y}$$

Formulation for optimum design of a cantilever beam. Thus the optimization problem is to find w and t to minimize the cost function of Eq. (s) subject to the eight inequality constraints of Eqs. (t)–(y). Note that the constraints of Eqs. (t)–(v) are nonlinear functions and others are linear functions of the design variables (the width-thickness ratio constraint in Eq. (w) has been transformed to the linear form). There are eight inequality constraints and no equality constraints for this problem. Note that each constraint depends on at least one design variable. Substituting various expressions, constraints in Eqs. (t)–(v) can be expressed explicitly in terms of the design variables, if desired. Or, we can keep them in terms of the intermediate variables I and Q and treat them as such in numerical calculations. Later in chapter: Optimum Design: Numerical Solution Process and Excel Solver, an example of design of a plate girder is described where some intermediate variables are explicitly treated as dependent variables in the formulation.

Using the quantities calculated in Eqs. (k)–(r), let us check the status of the constraints for the cantilever beam design problem at the trial design point w = 60 mm and t = 10 mm:

Bending stress constraint: $\sigma \le \sigma_a$; $\sigma = 1385$ N/mm^2, $\sigma_a = 165$ N/mm^2; ∴ violated
Shear stress constraint: $\tau \le \tau_a$; $\tau = 21.93$ N/mm^2, $\tau_a = 90$ N/mm^2; ∴ satisfied
Deflection constraint: $q \le q_a$; $q = 262.73$ mm, $q_a = 10$ mm; ∴ violated
Width–thickness restriction: $\frac{w}{t} \le 8$; $\frac{w}{t} = \frac{60}{10} = 6$; ∴ satisfied

In addition, the width w is at its allowed minimum value and the thickness t is within its allowed values as given in Eqs. (x) and (y). This trial design violates bending stress and deflection constraints and therefore it is not a feasible design for the problem.

2.2 DESIGN OF A CAN

Step 1: *Project/problem description.* The purpose of this project is to design a can, shown in Fig. 2.3, to hold *at least* 400 mL of liquid (1 mL = 1 cm^3), as well as to meet other design requirements. The cans will be produced in the billions, so it is desirable to minimize their manufacturing costs. Since cost can be directly related to the surface area of the sheet metal

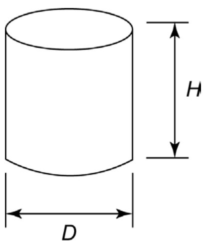

FIGURE 2.3 A can.

used, it is reasonable to minimize the amount of sheet metal required. Fabrication, handling, aesthetics, and shipping considerations impose the following restrictions on the size of the can: The diameter should be no more than 8 cm and no less than 3.5 cm, whereas the height should be no more than 18 cm and no less than 8 cm.

Step 2: *Data and information collection.* Data for the problem are given in the project statement.

Step 3: *Definition of design variable.* The two design variables are defined as

D = diameter of the can, cm
H = height of the can, cm

Step 4: *Optimization criterion.* The design objective is to minimize the total surface area S of the sheet metal for the three parts of the cylindrical can: the surface area of the cylinder (circumference × height) and the surface area of the two ends. Therefore, the optimization criterion, or *cost function* (the total area of sheet metal), is given as

$$S = \pi DH + 2\left(\frac{\pi}{4}D^2\right), \text{cm}^2 \tag{a}$$

Step 5: *Formulation of constraints.* The first constraint is that the can must hold *at least* 400 cm^3 of fluid, which is written as

$$\frac{\pi}{4}D^2H \geq 400, \text{cm}^3 \tag{b}$$

If it had been stated that "the can must hold 400 mL of fluid," then the preceding volume constraint would be an equality. The other constraints on the size of the can are

$$\begin{aligned} 3.5 \leq D \leq 8, \text{cm} \\ 8 \leq H \leq 18, \text{cm} \end{aligned} \tag{c}$$

The explicit constraints on design variables in Eq. (c) have many different names in the literature, such as *side constraints*, *technological constraints*, *simple bounds*, *sizing constraints*, and *upper and lower limits on the design variables*. Note that for the present problem there are really four constraints in Eq. (c). Thus, the problem has two design variables and a total of five inequality constraints. Note also that the cost function and the first constraint are nonlinear in variables; the remaining constraints are linear.

Mathematical formulation. Thus the optimization problem for design of a tank is to determine the design variables D and H to minimize the cost function in Eq. (a) subject to the five inequalities in Eqs. (b) and (c).

2.3 INSULATED SPHERICAL TANK DESIGN

Step 1: *Project/problem description.* The goal of this project is to choose an insulation thickness t to minimize the life-cycle cooling cost for a spherical tank. The cooling costs include installing and running the refrigeration equipment, and installing the insulation. Assume a 10-year life, a 10% annual interest rate, and no salvage value. The tank has already been designed having r (m) as its radius.

Step 2: *Data and information collection.* To formulate this design optimization problem, we need some data and analysis expressions. To calculate the volume of the insulation material, we require the surface area of the spherical tank, which is given as

$$A = 4\pi r^2,\ \text{m}^2 \tag{a}$$

To calculate the capacity of the refrigeration equipment and the cost of its operation, we need to calculate the annual heat gain G (Watt-hours), which is given as

$$G = \frac{(365)(24)(\Delta T)A}{c_1 t},\ \text{Wh} \tag{b}$$

where ΔT is the average difference between the internal and external temperatures in Kelvin, c_1 is the thermal resistivity per unit thickness in Kelvin-meter per Watt, and t is the insulation thickness in meters. ΔT can be estimated from the historical data for temperatures in the region in which the tank is to be used. Let c_2 = the insulation cost per cubic meter (\$/m^3), c_3 = the cost of the refrigeration equipment per Watt-hour of capacity (\$/Wh), and c_4 = the annual cost of running the refrigeration equipment per Watt-hour (\$/Wh).

Step 3: *Definition of design variables.* Only one design variable is identified for this problem:

t = insulation thickness, m.

Step 4: *Optimization criterion.* The goal is to minimize the life-cycle cooling cost of refrigeration for the spherical tank over 10 years. The life-cycle cost has three components: insulation, refrigeration equipment, and operations for 10 years. Once the annual operations cost has been converted to the present cost, the total cost is given as

$$Cost = c_2 At + c_3 G + c_4 G[uspwf\,(0.1, 10)] \tag{c}$$

where $uspwf\,(0.1, 10) = 6.14457$ is the uniform series present worth factor, calculated using the equation

$$uspwf\,(i, n) = \frac{1}{i}[1 - (1 - i)^{-n}] \tag{d}$$

where i is the rate of return per dollar per period and n is the number of periods. Note that to calculate the volume of the insulation as At, it is assumed that the insulation thickness is much smaller than the radius of the spherical tank; that is, $t \ll r$.

Step 5: *Formulation of constraints.* Although no constraints are indicated in the problem statement, it is important to require that the insulation thickness be nonnegative (ie, $t \geq 0$). Although this may appear obvious, it is important to include the constraint explicitly in the mathematical formulation of the problem. Without its explicit inclusion, the mathematics of optimization may assign negative values to thickness, which is, of course, meaningless. Note also that in reality t cannot be zero because it appears in the denominator of the expression for G. Therefore, the constraint should really be expressed as $t > 0$. However, *strict inequalities* cannot be treated mathematically or numerically in the solution process because they give an open feasible set. We must allow the possibility of satisfying inequalities as equalities; that is, we must allow the possibility that $t = 0$ in the solution process. Therefore, a more realistic constraint is $t \geq t_{min}$, where t_{min} is the smallest insulation thickness available on the market.

EXAMPLE 2.6 FORMULATION OF THE SPHERICAL TANK PROBLEM WITH INTERMEDIATE VARIABLES

A summary of the problem formulation for the design optimization of insulation for a spherical tank with intermediate variables is as follows:

Specified data: r, ΔT, c_1, c_2, c_3, c_4, $t_{\min}$
Design variable: t, m
Intermediate variables: A, m; G, Watt-hours

$$A = 4\pi r^2 \qquad G = \frac{(365)(24)(\Delta T)A}{c_1 t} \tag{e}$$

Cost function: Minimize the life-cycle cooling cost of refrigeration of the spherical tank,

$$Cost = c_2 At + c_3 G + 6.14457 c_4 G, \$ \tag{f}$$

Constraint:

$$t \ge t_{\min} \tag{g}$$

Note that A and G are also treated as design variables in this formulation. However, A must be assigned a fixed numerical value since r has already been determined, and the expression for G in Eq. (e) must be treated as an equality constraint.

Mathematical formulation. Thus the optimization problem for design of an insulated spherical tank is to determine the design variables t and G to minimize the cost function of Eq. (f) subject to the equality constraint in Eq. (e) and the inequality constraint on thickness in Eq. (g).

EXAMPLE 2.7 FORMULATION OF THE SPHERICAL TANK PROBLEM WITH THE DESIGN VARIABLE ONLY

Following is a summary of the problem formulation for the design optimization of insulation for a spherical tank in terms of the design variable only:

Specified data: r, ΔT, c_1, c_2, c_3, c_4, $t_{\min}$
Design variable: t, m
Cost function: Minimize the life-cycle cooling cost of refrigeration of the spherical tank,

$$Cost = at + \frac{b}{t},\ a = 4c_2\pi r^2, \qquad b = \frac{(c_3 + 6.14457 c_4)}{c_1}(365)(24)(\Delta T)(4\pi r^2) \tag{h}$$

Constraint:

$$t \geq t_{\min} \tag{i}$$

Mathematical formulation. Thus the optimization problem for design of an insulated spherical tank is to determine the design variable t to minimize the cost function of Eq. (h) subject to the minimum thickness constraint in Eq. (i).

2.4 SAWMILL OPERATION

Step 1: *Project/problem description*. A company owns two sawmills and two forests. Table 2.2 shows the capacity of each of the mills (logs/day) and the distances between the forests and the mills (km). Each forest can yield up to 200 logs/day for the duration of the project, and the cost to transport the logs is estimated at $10/km/log. At least 300 logs are needed daily. The goal is to minimize the total daily cost of transporting the logs and meet the constraints on the demand and the capacity of the mills.

Step 2: *Data and information collection*. Data are given in Table 2.2 and in the problem statement.

Step 3: *Definition of design variables*. The design problem is to determine how many logs to ship from Forest i to Mill j, as shown in Fig. 2.4. Therefore, the design variables are identified and defined as follows:

x_1 = number of logs shipped from Forest 1 to Mill A
x_2 = number of logs shipped from Forest 2 to Mill A
x_3 = number of logs shipped from Forest 1 to Mill B
x_4 = number of logs shipped from Forest 2 to Mill B

TABLE 2.2 Data for Sawmills

Mill	Distance from Forest 1	Distance from Forest 2	Mill capacity per day
A	24.0 km	20.5 km	240 logs
B	17.2 km	18.0 km	300 logs

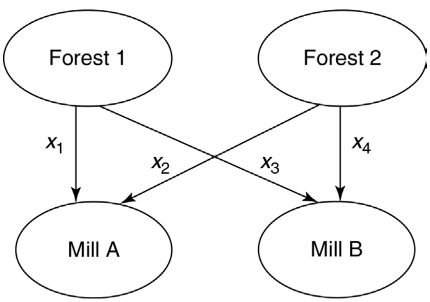

FIGURE 2.4 **Sawmill operation.**

Note that if we assign numerical values to these variables, an operational plan for the project is specified and the cost of daily log transportation can be calculated; that is, they are independent design variables. The selected design may or may not satisfy all the constraints.

Step 4: *Optimization criterion.* The design objective is to minimize the daily cost of transporting the logs to the mills. The cost of transportation, which depends on the distance between the forests and the mills given in Table 2.2, is

$$\begin{aligned} Cost &= 24(10)x_1 + 20.5(10)x_2 + 17.2(10)x_3 + 18(10)x_4 \\ &= 240.0x_1 + 205.0x_2 + 172.0x_3 + 180.0x_4 \end{aligned} \tag{a}$$

Step 5: *Formulation of constraints.* The constraints for the problem are based on mill capacity and forest yield:

$$\begin{aligned} &x_1 + x_2 \le 240 \text{ (Mill A Capacity)} \\ &x_3 + x_4 \le 300 \text{ (Mill B Capacity)} \\ &x_1 + x_3 \le 200 \text{ (Forest 1 yield)} \\ &x_2 + x_4 \le 200 \text{ (Forest 2 yield)} \end{aligned} \tag{b}$$

The constraint on the number of logs needed for each day is expressed as

$$x_1 + x_2 + x_3 + x_4 \ge 300 \text{ (demand for logs)} \tag{c}$$

For a realistic problem formulation, all design variables must be non-negative; that is,

$$x_i \ge 0; \quad i = 1 \text{ to } 4 \tag{d}$$

Mathematical formulation. The problem has four design variables, five inequality constraints, and four nonnegativity constraints on the design variables. The optimization problem is to determine the design variables x_1–x_4 to minimize the cost function in Eq. (a) subject to the constraints in Eqs. (b)–(d). Note that all problem functions are linear in design variables, so this is a *linear programming problem*. Note also that for a meaningful solution, all design variables must have *integer* values. Such problems are called *integer-programming problems* and require special solution methods. Some such methods are discussed in chapter: Discrete Variable Optimum Design Concepts and Methods.

It is also noted that the problem of sawmill operation falls into a class known as *transportation problems*. For such problems, we would like to ship items from several distribution centers to several retail stores to meet their demand at a minimum cost of transportation. Special methods have been developed to solve this class of problems.

2.5 DESIGN OF A TWO-BAR BRACKET

Step 1: *Project/problem description.* The objective of this project is to design a two-bar bracket (shown in Fig. 2.5) to support a load W without structural failure. The load is applied at an angle θ, which is between 0 and 90°, h is the height, and s is the bracket's base width. The bracket will be produced in large quantities. It has also been determined that its total cost

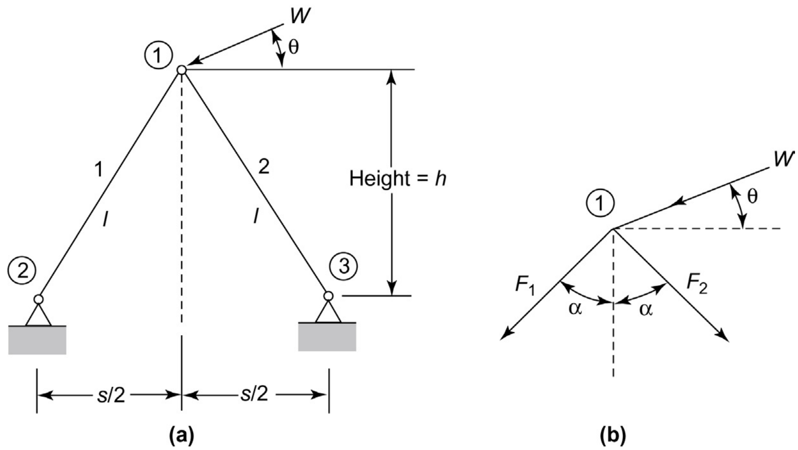

FIGURE 2.5 **Two-bar bracket.** (a) Structure and (b) free-body diagram for node 1.

(material, fabrication, maintenance, and so on) is directly related to the size of the two bars. Thus, the design objective is to minimize the total mass of the bracket while satisfying performance, fabrication, and space limitations.

Step 2: *Data and information collection.* First, the load W and its angle of application θ need to be specified. Since the bracket may be used in several applications, it may not be possible to specify just one angle for W. It is possible to formulate the design optimization problem such that a range is specified for angle θ (ie, load W may be applied at any angle within that specified range). In this case, the formulation will be slightly more complex because performance requirements will need to be satisfied for each angle of application. However, in the present formulation, it is assumed that angle θ is specified.

Second, the material to be used for the bars must be specified because the material properties are needed to formulate the optimization criterion and performance requirements. Whether the two bars are to be fabricated using the same material also needs to be determined. In the present formulation, it is assumed that they are, although it may be prudent to assume otherwise for some advanced applications. In addition, we need to determine the fabrication and space limitations for the bracket (eg, on the size of the bars, height, and base width).

In formulating the design problem, we also need to define *structural performance* more precisely. Forces F_1 and F_2 carried by bars 1 and 2, respectively, can be used to define failure conditions for the bars. To compute these forces, we use the principle of *static equilibrium.* Using the *free-body diagram* for node 1 (shown in Fig. 2.5b), equilibrium of forces in the horizontal and vertical directions gives

$$\begin{aligned} -F_1 \sin\alpha + F_2 \sin\alpha &= W \cos\theta \\ -F_1 \cos\alpha - F_2 \cos\alpha &= W \sin\theta \end{aligned} \qquad \text{(a)}$$

From the geometry of Fig. 2.5, $\sin\alpha = 0.5\, s/l$ and $\cos\alpha = h/l$, where l is the length of members given as $l = \sqrt{h + (0.5s)^2}$. Note that F_1 and F_2 are shown as tensile forces in the free-body diagram. The solution to Eq. (a) will determine the magnitude and direction of the forces. In addition, the *tensile force will be taken as positive*. Thus, the bar will be in compression if the force carried by it has negative value. By solving the two equations simultaneously for the unknowns F_1 and F_2, we obtain

$$F_1 = -0.5Wl\left[\frac{\sin\theta}{h} + \frac{2\cos\theta}{s}\right]$$
$$F_2 = -0.5Wl\left[\frac{\sin\theta}{h} - \frac{2\cos\theta}{s}\right] \quad \text{(b)}$$

To avoid bar failure due to overstressing, we need to calculate bar stress. If we know the force carried by a bar, then the stress σ can be calculated as the force divided by the bar's cross-sectional area (stress = force/area). The SI unit for stress is Newton/meter^2 (N/m^2), also called Pascal (Pa), whereas the US–British unit is pound/in^2 (written as psi). The expression for the cross-sectional area depends on the cross-sectional shape used for the bars and selected design variables. Therefore, a structural shape for the bars and associated design variables must be selected. This is illustrated later in the formulation process.

In addition to analysis equations, we need to define the properties of the selected material. Several formulations for optimum design of the bracket are possible depending on the application's requirements. To illustrate, a material with known properties is assumed for the bracket. However, the structure can be optimized using other materials along with their associated fabrication costs. Solutions can then be compared to select the best possible material for the structure.

For the selected material, let ρ be the mass density and $\sigma_a > 0$ be the allowable design stress. As a performance requirement, it is assumed that if the stress exceeds this allowable value, the bar is considered to have failed. The *allowable stress* is defined as the material failure stress (a property of the material) divided by a factor of safety greater than one. In addition, it is assumed that the allowable stress is calculated in such a way that the buckling failure of a bar in compression is avoided.

Step 3: *Definition of design variables.* Several sets of design variables may be identified for the two-bar structure. The height h and span s can be treated as design variables in the initial formulation. Later, they may be assigned numerical values, if desired, to eliminate them from the formulation. Other design variables will depend on the cross-sectional shape of bars 1 and 2. Several cross-sectional shapes are possible, as shown in Fig. 2.6, where design variables for each shape are also identified.

Note that for many cross-sectional shapes, different design variables can be selected. For example, in the case of the circular tube in Fig. 2.6a, the outer diameter d_o and the ratio between the inner and outer diameters $r = d_i/d_o$ may be selected as the design variables. Or d_o and d_i may be selected. However, it is not desirable to designate d_o, d_i, and r as the design variables because they are not independent of each other. If they are selected, then a relationship between them must be specified as an equality constraint. Similar remarks can be made for the design variables associated with other cross-sections, also shown in Fig. 2.6.

As an example of problem formulation, consider the design of a bracket with hollow circular tubes as members, as shown in Fig. 2.6a. The inner and outer diameters d_i and d_o and wall

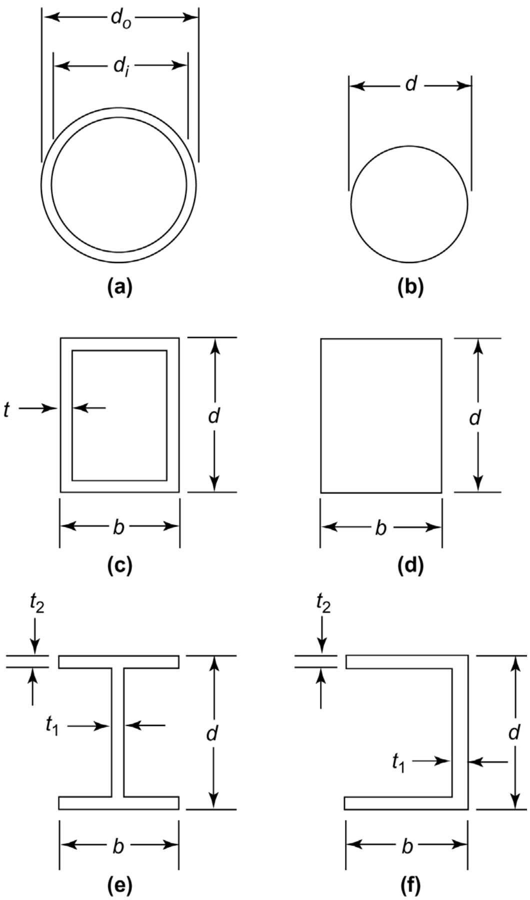

FIGURE 2.6 **Bar cross-sectional shapes.** (a) Circular tube; (b) solid circular; (c) rectangular tube; (d) solid rectangular; (e) I-section; (f) channel section.

thickness t may be identified as the design variables, although they are not all independent of each other. For example, we cannot specify $d_i = 10$, $d_o = 12$, and $t = 2$ because it violates the physical condition $t = 0.5(d_o - d_i)$. Therefore, if we formulate the problem with d_i, d_o, and t as design variables, we must also impose the constraint $t = 0.5(d_o - d_i)$. To illustrate a formulation of the problem, let the design variables be defined as

x_1 = height h of the bracket
x_2 = span s of the bracket
x_3 = outer diameter of bar 1
x_4 = inner diameter of bar 1
x_5 = outer diameter of bar 2
x_6 = inner diameter of bar 2

In terms of these variables, the cross-sectional areas A_1 and A_2 of bars 1 and 2 are given as

$$A_1 = \frac{\pi}{4}(x_3^2 - x_4^2);\ A_2 = \frac{\pi}{4}(x_5^2 - x_6^2) \tag{c}$$

Once the problem is formulated in terms of the six selected design variables, it is always possible to modify it to meet more specialized needs. For example, the height x_1 may be assigned a fixed numerical value, thus eliminating it from the problem formulation. In addition, complete symmetry of the structure may be required to make its fabrication easier; that is, it may be necessary for the two bars to have the same cross-section, size, and material. In such a case, we set $x_3 = x_5$ and $x_4 = x_6$ in all expressions of the problem formulation. Such modifications are left as exercises.

Step 4: *Optimization criterion.* The structure's mass is identified as the objective function in the problem statement. Since it is to be minimized, it is called the *cost function* for the problem. An expression for the mass is determined by the cross-sectional shape of the bars and associated design variables. For the hollow circular tubes and selected design variables, the total mass of the structure is calculated as (density × material volume):

$$Mass = \rho[l(A_1 + A_2)] = \left[\rho\sqrt{x_1^2 + (0.5x_2)^2}\right]\frac{\pi}{4}(x_3^2 - x_4^2 + x_5^2 - x_6^2) \tag{d}$$

Note that if the outer diameter and the ratio between the inner and outer diameters are selected as design variables, the form of the mass function changes. Thus, the *final form* depends on the design variables selected for the problem.

Step 5: *Formulation of constraints.* It is important to include all constraints in the problem formulation because the final solution depends on them. For the two-bar structure, the constraints are on the stress in the bars and on the design variables themselves. These constraints will be formulated for hollow circular tubes using the previously defined design variables. They can be similarly formulated for other sets of design variables and cross-sectional shapes.

To avoid overstressing a bar, the calculated stress σ (tensile or compressive) must not exceed the material allowable stress $\sigma_a > 0$. The stresses σ_1 and σ_2 in the two bars are calculated as force/area:

$$\begin{aligned} \sigma_1 &= \frac{F_1}{A_1} \quad \text{(stress in bar 1)} \\ \sigma_2 &= \frac{F_2}{A_2} \quad \text{(stress in bar 2)} \end{aligned} \tag{e}$$

Note that to treat positive and negative stresses (tension and compression), we must use the absolute value of the calculated stress in writing the constraints (eg, $|\sigma| \le \sigma_a$). The absolute-value constraints can be treated by different approaches in optimization methods. Here we split each absolute-value constraint into two constraints. For example, the stress constraint for bar 1 is written as the following two constraints:

$$\begin{aligned} \sigma_1 &\le \sigma_a \text{ (tensile stress in bar 1)} \\ -\sigma_1 &\le \sigma_a \text{ (compressive stress in bar 1)} \end{aligned} \tag{f}$$

With this approach, the second constraint is satisfied automatically if bar 1 is in tension, and the first constraint is automatically satisfied if bar 1 is in compression. Similarly, the stress constraint for bar 2 is written as

$$\begin{aligned} &\sigma_2 \le \sigma_a \text{ (tensile stress in bar 2)} \\ &-\sigma_2 \le \sigma_a \text{ (compressive stress in bar 2)} \end{aligned} \tag{g}$$

Finally, to impose fabrication and space limitations, constraints on the design variables are imposed as

$$x_{iL} \le x_i \le x_{iU}; \quad i = 1 \text{ to } 6 \tag{h}$$

where x_{iL} and x_{iU} are the minimum and maximum allowed values for the ith design variable. Their numerical values must be specified before the problem can be solved.

Note that the expression for bar stress changes if different design variables are chosen for circular tubes, or if a different cross-sectional shape is chosen for the bars. For example, inner and outer radii, mean radius and wall thickness, or outside diameter and the ratio of inside to outside diameter as design variables will all produce different expressions for the cross-sectional areas and stresses. *These results show that the choice of design variables greatly influences the problem formulation.*

Note also that we had to first *analyze* the structure (calculate its response to given inputs) to write the constraints properly. It was only after we had calculated the forces in the bars that we were able to write the constraints. This is an important step in any engineering design problem formulation: *We must be able to analyze the system before we can formulate the design optimization problem.*

In the following examples, we summarize two formulations of the problem. The first uses several intermediate variables, which is useful when the problem is transcribed into a computer program. Because this formulation involves simpler expressions of various quantities, it is easier to write and debug a computer program. In the second formulation, all intermediate variables are eliminated to obtain the formulation exclusively in terms of design variables. This formulation has slightly more complex expressions. It is important to note that the second formulation may not be possible for all applications because some problem functions may only be implicit functions of the design variables. One such formulation is presented in chapter: Practical Applications of Optimization.

EXAMPLE 2.8 FORMULATION OF THE TWO-BAR BRACKET PROBLEM WITH INTERMEDIATE VARIABLES

A summary of the problem formulation for optimum design of the two-bar bracket using intermediate variables is as follows:

Specified data: W, θ, $\sigma_a > 0$, x_{iL} and x_{iU}, $i = 1$ to 6
Design variables: x_1, x_2, x_3, x_4, x_5, x_6
Intermediate variables:
Bar cross-sectional areas:

$$A_1 = \frac{\pi}{4}(x_3^2 - x_4^2);\ A_2 = \frac{\pi}{4}(x_5^2 - x_6^2) \tag{a}$$

Length of bars:

$$l = \sqrt{x_1^2 + (0.5x_2)^2} \tag{b}$$

Forces in bars:

$$\begin{aligned} F_1 &= -0.5Wl\left[\frac{\sin\theta}{x_1} + \frac{2\cos\theta}{x_2}\right] \\ F_2 &= -0.5Wl\left[\frac{\sin\theta}{x_1} - \frac{2\cos\theta}{x_2}\right] \end{aligned} \tag{c}$$

Bar stresses:

$$\sigma_1 = \frac{F_1}{A_1};\quad \sigma_2 = \frac{F_2}{A_2} \tag{d}$$

Cost function: Minimize the total mass of the bars,

$$Mass = \rho l(A_1 + A_2) \tag{e}$$

Constraints:
Bar stress:

$$-\sigma_1 \le \sigma_a;\ \sigma_1 \le \sigma_a;\ -\sigma_2 \le \sigma_a;\ \sigma_2 \le \sigma_a \tag{f}$$

Design variable limits:

$$x_{iL} \le x_i \le x_{iU};\quad i = 1 \text{ to } 6 \tag{g}$$

Mathematical formulation. Thus when the intermediate variables are also treated as design variables, the optimization problem becomes: determine the design variables A_1, A_2, l, F_1, F_2, σ_1, σ_2, and x_1–x_6 to minimize the cost function in Eq. (e) subject to 7 equality constraints in Eqs. (a)–(d) and 16 inequality constraints in Eqs. (f) and (g).

EXAMPLE 2.9 FORMULATION OF THE TWO-BAR BRACKET WITH DESIGN VARIABLES ONLY

A summary of the problem formulation for optimum design of the two-bar bracket in terms of design variables only is obtained by eliminating the intermediate variables from all the expressions as follows:

Specified data: W, θ, $\sigma_a > 0$, x_{iL} and x_{iU}, $i = 1$ to 6
Design variables: x_1, x_2, x_3, x_4, x_5, x_6

Cost function: Minimize total mass of the bars,

$$Mass = \frac{\pi\rho}{4}\sqrt{x_1^2 + (0.5x_2)^2}\,(x_3^2 - x_4^2 + x_5^2 - x_6^2) \tag{a}$$

Constraints:
Bar stress:

$$\frac{2W\sqrt{x_1^2 + (0.5x_2)^2}}{\pi(x_3^2 - x_4^2)}\left[\frac{\sin\theta}{x_1} + \frac{2\cos\theta}{x_2}\right] \le \sigma_a \tag{b}$$

$$\frac{-2W\sqrt{x_1^2 + (0.5x_2)^2}}{\pi(x_3^2 - x_4^2)}\left[\frac{\sin\theta}{x_1} + \frac{2\cos\theta}{x_2}\right] \le \sigma_a \tag{c}$$

$$\frac{2W\sqrt{x_1^2 + (0.5x_2)^2}}{\pi(x_5^2 - x_6^2)}\left[\frac{\sin\theta}{x_1} - \frac{2\cos\theta}{x_2}\right] \le \sigma_a \tag{d}$$

$$\frac{-2W\sqrt{x_1^2 + (0.5x_2)^2}}{\pi(x_5^2 - x_6^2)}\left[\frac{\sin\theta}{x_1} - \frac{2\cos\theta}{x_2}\right] \le \sigma_a \tag{e}$$

Design variable limits:

$$x_{iL} \le x_i \le x_{iU}; \quad i = 1 \text{ to } 6 \tag{f}$$

Mathematical formulation. Thus the optimization problem is to determine the design variables x_1–x_6 to minimize the cost function in Eq. (a) subject to 16 inequality constraints in Eqs. (b)–(f).

It is important to note that the intermediate variables can be treated as design variables in the formulation of the optimization problem. Usually this results in a simpler set of equations in the formulation but at the cost of additional equality constraints.

2.6 DESIGN OF A CABINET

Step 1: *Project/problem description.* A cabinet is assembled from components C_1, C_2, and C_3. Each cabinet requires 8 C_1, 5 C_2, and 15 C_3 components. The assembly of C_1 requires either 5 bolts or 5 rivets, whereas C_2 requires 6 bolts or 6 rivets, and C_3 requires 3 bolts or 3 rivets. The cost of installing a bolt, including the cost of the bolt itself, is \$0.70 for C_1, \$1.00 for C_2, and \$0.60 for C_3. Similarly, riveting costs are \$0.60 for C_1, \$0.80 for C_2, and \$1.00 for C_3. Bolting and riveting capacities per day are 6000 and 8000, respectively. To minimize the cost for the 100 cabinets that must be assembled each day, we wish to determine the number of components to be bolted and riveted (after Siddall, 1972).

Step 2: *Data and information collection.* All data for the problem are given in the project statement.

This problem can be formulated in several different ways depending on the assumptions made and the definition of the design variables. Three formulations are presented, and for each one, the design variables are identified and expressions for the cost and constraint functions are derived; that is, steps 3–5 are presented.

2.6.1 Formulation 1 for Cabinet Design

Step 3: *Definition of design variables.* In the first formulation, the following design variables are identified for 100 cabinets:

x_1 = number of C_1 to be bolted for all 100 cabinets
x_2 = number of C_1 to be riveted for all 100 cabinets
x_3 = number of C_2 to be bolted for all 100 cabinets
x_4 = number of C_2 to be riveted for all 100 cabinets
x_5 = number of C_3 to be bolted for all 100 cabinets
x_6 = number of C_3 to be riveted for all 100 cabinets

Step 4: *Optimization criterion.* The design objective is to minimize the total cost of cabinet fabrication, which is obtained from the specified costs for bolting and riveting each component:

$$\begin{aligned} Cost &= 0.70(5)x_1 + 0.60(5)x_2 + 1.00(6)x_3 + 0.80(6)x_4 + 0.60x_5 + 1.00(3)x_6 \\ &= 3.5x_1 + 3.0x_2 + 6.0x_3 + 4.8x_4 + 1.8x_5 + 3.0x_6 \end{aligned} \tag{a}$$

Step 5: *Formulation of constraints.* The constraints for the problem consist of riveting and bolting capacities and the number of cabinets fabricated each day. Since 100 cabinets must be fabricated, the required numbers of C_1, C_2, and C_3 are given in the following constraints:

$$\begin{aligned} &\text{Number of } C_1 \text{ used must be } 8\times 100: x_1 + x_2 = 8\times 100 \\ &\text{Number of } C_2 \text{ used must be } 5\times 100: x_3 + x_4 = 5\times 100 \\ &\text{Number of } C_3 \text{ used must be } 15\times 100: x_5 + x_6 = 15\times 100 \end{aligned} \tag{b}$$

Bolting and riveting capacities must not be exceeded. Thus,

$$\begin{aligned} &\text{Bolting capacity: } 5x_1 + 6x_3 + 3x_5 \le 6000 \\ &\text{Riveting capacity: } 5x_2 + 6x_4 + 3x_6 \le 8000 \end{aligned} \tag{c}$$

Finally, all design variables must be nonnegative for a meaningful solution:

$$x_i \ge 0; i = 1 \text{ to } 6 \tag{d}$$

Mathematical formulation. Thus, the optimization problem is to determine six design variables x_1 to x_6 subject to three equality constraints, and eight inequality constraints in Eqs. (b)–(d).

2.6.2 Formulation 2 for Cabinet Design

Step 3: *Definition of design variables.* If we relax the constraint that each component must be bolted or riveted, then the following design variables can be defined:

x_1 = total number of bolts required for all C_1
x_2 = total number of bolts required for all C_2
x_3 = total number of bolts required for all C_3

x_4 = total number of rivets required for all C_1
x_5 = total number of rivets required for all C_2
x_6 = total number of rivets required for all C_3

Step 4: *Optimization criterion.* The objective is still to minimize the total cost of fabricating 100 cabinets, given as

$$Cost = 0.70x_1 + 1.00x_2 + 0.60x_3 + 0.60x_4 + 0.80x_5 + 1.00x_6, \$ \tag{e}$$

Step 5: *Formulation of constraints.* Since 100 cabinets must be built every day, it will be necessary to have 800 C_1, 500 C_2, and 1500 C_3 components. The total number of bolts and rivets needed for all C_1, C_2, and C_3 components is indicated by the following equality constraints:

$$\begin{aligned} &\text{Bolts and rivets needed for } C_1: x_1 + x_4 = 5 \times 800 \\ &\text{Bolts and rivets needed for } C_2: x_2 + x_5 = 6 \times 500 \\ &\text{Bolts and rivets needed for } C_3: x_3 + x_6 = 3 \times 1500 \end{aligned} \tag{f}$$

Bolting and riveting capacities must not be exceeded. Thus,

$$\begin{aligned} &\text{Bolting capacity: } x_1 + x_2 + x_3 \le 6000 \\ &\text{Riveting capacity: } x_4 + x_5 + x_6 \le 8000 \end{aligned} \tag{g}$$

Finally, all design variables must be non-negative:

$$x_i \ge 0; i = 1 \text{ to } 6 \tag{h}$$

Mathematical formulation. Thus, the optimization problem is to determine six design variables x_1–x_6 subject to three equality constraints and eight inequality constraints in Eqs. (g) and (h). After an optimum solution has been obtained, we can decide how many components to bolt and how many to rivet.

2.6.3 Formulation 3 for Cabinet Design

Step 3: *Definition of design variables.* Another formulation of the problem is possible if we require that all cabinets be identical. The following design variables can be identified:

x_1 = number of C_1 to be bolted on one cabinet
x_2 = number of C_1 to be riveted on one cabinet
x_3 = number of C_2 to be bolted on one cabinet
x_4 = number of C_2 to be riveted on one cabinet
x_5 = number of C_3 to be bolted on one cabinet
x_6 = number of C_3 to be riveted on one cabinet

Step 4: *Optimization criterion.* With these design variables, the cost of fabricating 100 cabinets each day is given as

$$\begin{aligned} Cost &= 100[0.70(5)x_1 + 0.60(5)x_2 + 1.00(6)x_3 + 0.80(6)x_4 + 0.60x_5 + 1.00(3)x_6] \\ &= 350x_1 + 300x_2 + 600x_3 + 480x_4 + 180x_5 + 300x_6 \end{aligned} \tag{i}$$

Step 5: *Formulation of constraints.* Since each cabinet needs 8 C_1, 5 C_2, and 15 C_3 components, the following equality constraints can be identified:

$$\begin{aligned} x_1 + x_2 &= 8 \ \ \text{(number of } C_1 \text{ needed)} \\ x_3 + x_4 &= 5 \ \ \text{(number of } C_2 \text{ needed)} \\ x_5 + x_6 &= 15 \ \ \text{(number of } C_3 \text{ needed)} \end{aligned} \tag{j}$$

Constraints on the capacity to rivet and bolt are expressed as the following inequalities:

$$\begin{aligned} (5x_1 + 6x_3 + 3x_5)100 &\le 6000 \ \ \text{(bolting capacity)} \\ (5x_2 + 6x_4 + 3x_6)100 &\le 8000 \ \ \text{(riveting capacity)} \end{aligned} \tag{k}$$

Finally, all design variables must be non-negative:

$$x_i \ge; \quad i = 1 \text{ to } 6 \tag{l}$$

Mathematical formulation. Thus, the optimization problem is to determine six design variables x_1–x_6 subject to three equality constraints, and eight inequality constraints in Eqs. (j) and (l).

The following points are noted for the three formulations:

1. Because cost and constraint functions are *linear* in all three formulations, they are linear programming problems. It is conceivable that each formulation will yield a different optimum solution. After solving the problems, the designer can select the best strategy for fabricating cabinets.
2. All formulations have *three equality constraints*, each involving two design variables. Using these constraints, we can eliminate three variables from the problem and thus reduce its dimension. This is desirable from a computational standpoint because the number of variables and constraints is reduced. However, because the elimination of variables is not possible for many complex problems, we must develop and use methods to treat both equality and inequality constraints.
3. For a meaningful solution for these formulations, all design variables must have integer values. These are called *integer programming problems*. Some numerical methods to treat this class of problem are discussed in chapter: Discrete Variable Optimum Design Concepts and Methods.

2.7 MINIMUM-WEIGHT TUBULAR COLUMN DESIGN

Step 1: *Project/problem description.* Straight columns are used as structural elements in civil, mechanical, aerospace, agricultural, and automotive structures. Many such applications can be observed in daily life, for example, a street light pole, a traffic light post, a flagpole, a water tower support, a highway signpost, a power transmission pole. It is important to optimize the design of a straight column since it may be mass-produced. The objective of this project is to design a minimum-mass *tubular* column of length l supporting a load P without buckling or overstressing. The column is fixed at the base and free at the top, as shown in Fig. 2.7. This type of structure is called a cantilever column.

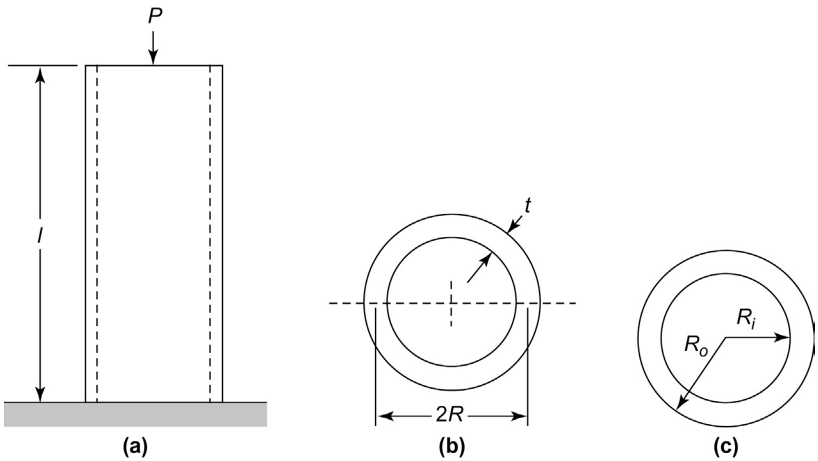

FIGURE 2.7 (a) Tubular column; (b) formulation 1 design variables; (c) formulation 2 design variables.

Step 2: *Data and information collection.* The buckling load (also called the critical load) for a cantilever column is given as

$$P_{\mathrm{cr}} = \frac{\pi^2 EI}{4l^2} \tag{a}$$

The buckling load formula for a column with other support conditions is different from this formula (Crandall et al., 2012). Here, I is the moment of inertia for the cross-section of the column and E is the material property, called the modulus of elasticity (Young's modulus). Note that the buckling load depends on the design of the column (ie, the moment of inertia I). It imposes a limit on the applied load; that is, the column fails if the applied load exceeds the buckling load. The material stress σ for the column is defined as P/A, where A is the cross-sectional area of the column. The material allowable stress under the axial load is σ_a, and the material mass density is ρ (mass per unit volume).

A cross-section of the tubular column is shown in Fig. 2.7. Many formulations for the design problem are possible depending on how the design variables are defined. Two such formulations are described here.

2.7.1 Formulation 1 for Column Design

Step 3: *Definition of design variables.* For the first formulation, the following design variables are defined:

R = mean radius of the column
t = wall thickness

Assuming that the column wall is thin ($R \gg t$), the material cross-sectional area and moment of inertia are

$$A = 2\pi Rt; \; I = \pi R^3 t \tag{b}$$

Step 4: *Optimization criterion.* The total mass of the column to be minimized is given as

$$Mass = \rho(lA) = 2\rho l\pi Rt \tag{c}$$

Step 5: *Formulation of constraints.* The first constraint is that the stress (P/A) should not exceed the material allowable stress σ_a to avoid material failure. This is expressed as the inequality $\sigma \le \sigma_a$. Replacing σ with P/A and then substituting for A, we obtain

$$\frac{P}{2\pi Rt} \le \sigma_a \tag{d}$$

The column should not buckle under the applied load P, which implies that the applied load should not exceed the buckling load (ie, $P \le P_{cr}$). Using the given expression for the buckling load in Eq. (a) and substituting for I, we obtain

$$P \le \frac{\pi^3 ER^3 t}{4l^2} \tag{e}$$

Finally, the design variables R and t must be within the specified minimum (R_{min} and t_{min}) and maximum values (R_{max} and t_{max}):

$$R_{min} \le R \le R_{max}\,;\; t_{min} \le t \le t_{max} \tag{f}$$

Mathematical formulation. Thus the optimization problem is to determine the design variables R and t to minimize the cost function in Eq. (c) subject to six inequality constraints in Eqs. (d)–(f).

2.7.2 Formulation 2 for Column Design

Step 3: *Definition of design variables.* Another formulation of the design problem is possible if the following design variables are defined:

R_o = outer radius of the column
R_i = inner radius of the column

In terms of these design variables, the cross-sectional area A and the moment of inertia I are

$$A = \pi(R_o^2 - R_i^2);\; I = \frac{\pi}{4}(R_o^4 - R_i^4) \tag{g}$$

Step 4: *Optimization criterion.* Minimize the total mass of the column:

$$Mass = \rho(lA) = \pi\rho l(R_o^2 - R_i^2) \tag{h}$$

Step 5: *Formulation of the constraints.* The material crushing constraint is ($P/A \le \sigma_a$):

$$\frac{P}{\pi(R_o^2 - R_i^2)} \le \sigma_a \tag{i}$$

Using the expression for I, the buckling load constraint is ($P \le P_{cr}$):

$$P \le \frac{\pi^3 E}{16l^3}(R_o^4 - R_i^4) \tag{j}$$

Finally, the design variables R_o and R_i must be within specified minimum ($R_{o\,\min}$ and $R_{i\,\min}$) and maximum ($R_{o\,\max}$ and $R_{i\,\max}$) limits:

$$R_{o\,\min} \le R_o \le R_{o\,\max}; \quad R_{i\,\min} \le R_i \le R_{i\,\max} \tag{k}$$

When this problem is solved using a numerical method, a constraint $R_o > R_i$ must also be imposed. Otherwise, some methods may take the design to the point where $R_o < R_i$. This situation is not physically possible and must be explicitly excluded to numerically solve the design problem.

In addition to the foregoing constraints, local buckling of the column wall needs to be considered for both formulations. Local buckling can occur if the wall thickness becomes too small. This can be avoided if the ratio of mean radius to wall thickness is required to be smaller than a limiting value, that is,

$$\frac{(R_o + R_i)}{2(R_o - R_i)} \le k \text{ or } \frac{R}{t} \le k \tag{l}$$

where R is the mean radius, and k is a specified value that depends on Young's modulus and the yield stress of the material. For steel with E = 29,000 ksi and a yield stress of 50 ksi, k is given as 32 (AISC, 2011).

Mathematical formulation. Thus the optimization problem is to determine the design variables R_o and R_i to minimize the cost function in Eq. (h) subject to seven inequality constraints in Eqs. (i)–(l).

2.8 MINIMUM-COST CYLINDRICAL TANK DESIGN

Step 1: *Project/problem description.* Design a minimum-cost cylindrical tank closed at both ends to contain a fixed volume of fluid V. The cost is found to depend directly on the area of sheet metal used.

Step 2: *Data and information collection.* Let c be the dollar cost per unit area of the sheet metal. Other data are given in the project statement.

Step 3: *Definition of design variables.* The design variables for the problem are identified as

R = radius of the tank
H = height of the tank

Step 4: *Optimization criterion.* The cost function for the problem is the dollar cost of the sheet metal for the tank. Total surface area of the sheet metal consisting of the end plates and cylinder is given as

$$A = 2\pi R^2 + 2\pi RH \tag{a}$$

Therefore, the cost function for the problem is given as

$$f = c(2\pi R^2 + 2\pi RH) \tag{b}$$

Step 5: *Formulation of constraints.* The volume of the tank ($\pi R^2 H$) is required to be V. Therefore,

$$\pi R^2 H = V \tag{c}$$

Also, both of the design variables R and H must be within some minimum and maximum values:

$$R_{min} \le R \le R_{max}; \; H_{min} \le H \le H_{max} \tag{d}$$

Mathematical formulation. The optimization problem is to determine R and H to minimize the cost function in Eq. (b) subject to one equality constraint in Eq. (c) and four inequalities in Eq. (d). This problem is quite similar to the can problem discussed in Section 2.2. The only difference is in the volume constraint. There the constraint is an inequality and here it is an equality.

2.9 DESIGN OF COIL SPRINGS

Step 1: *Project/problem description.* Coil springs are used in numerous practical applications. Detailed methods for analyzing and designing such mechanical components have been developed over the years (eg, Spotts, 1953; Wahl, 1963; Haug and Arora, 1979; Budynas and Nisbett, 2014). The purpose of this project is to design a minimum-mass spring (shown in Fig. 2.8) to carry a given axial load (called a tension–compression spring) without material failure and while satisfying two performance requirements: the spring must deflect by at least Δ (in.) and the frequency of surge waves must not be less than ω_0 (Hz).

Step 2: *Data and information collection.* To formulate the problem of designing a coil spring, see the notation and data defined in Table 2.3.

The wire twists when the spring is subjected to a tensile or a compressive load. Therefore, shear stress needs to be calculated so that a constraint on it can be included in the formulation. In addition, surge wave frequency needs to be calculated. These and other design equations for the spring are given as

Load deflection equation:

$$P = K\delta \tag{a}$$

Spring constant, K:

$$K = \frac{d^4 G}{8D^3 N} \tag{b}$$

Shear stress, τ:

$$\tau = \frac{8kPD}{\pi d^3} \tag{c}$$

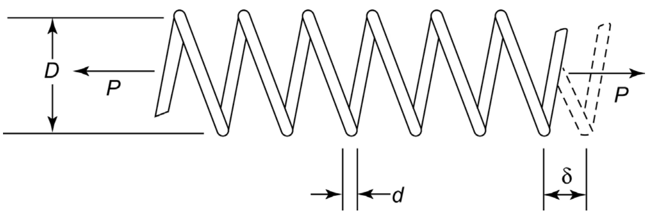

FIGURE 2.8 A coil spring.

TABLE 2.3 Information to Design a Coil Spring

Notation	Data
Deflection along the axis of spring	δ, in.
Mean coil diameter	D, in.
Wire diameter	d, in.
Number of active coils	N
Gravitational constant	$g = 386$ in./s^2
Frequency of surge waves	ω, Hz
Weight density of spring material	$\gamma = 0.285$ lb/in^3
Shear modulus	$G = (1.15 \times 10^7)$ lb/in^2
Mass density of material ($\rho = \gamma/g$)	$\rho = (7.38342 \times 10^{-4})$ lb-s^2/in^4
Allowable shear stress	$\tau_a = 80{,}000$ lb/in^2
Number of inactive coils	$Q = 2$
Applied load	$P = 10$ lb
Minimum spring deflection	$\Delta = 0.5$ in.
Lower limit on surge wave frequency	$\omega_0 = 100$ Hz
Limit on outer diameter of coil	$D_o = 1.5$ in.

Wahl stress concentration factor, k:

$$k = \frac{(4D - d)}{4(D - d)} + \frac{0.615d}{D} \tag{d}$$

Frequency of surge waves, ω:

$$\omega = \frac{d}{2\pi N D^2}\sqrt{\frac{G}{2\rho}} \tag{e}$$

The expression for the Wahl stress concentration factor k in Eq. (d) has been determined experimentally to account for unusually high stresses at certain points on the spring. These analysis equations are used to define the constraints.

Step 3: *Definition of design variables.* The three design variables for the problem are defined as

d = wire diameter, in
D = mean coil diameter, in
N = number of active coils, integer

Step 4: *Optimization criterion.* The problem is to *minimize the mass* of the spring, given as volume × mass density:

$$Mass = \frac{\pi}{4}d^2[(N+Q)\pi D]\rho = \frac{1}{4}(N+Q)\pi^2 D d^2 \rho \tag{f}$$

Step 5: *Formulation of constraints*

Deflection constraint. It is often a requirement that *deflection* under a load P be at least Δ. Therefore, the constraint is that the calculated deflection δ must be greater than or equal to a specified limit Δ. Such a constraint is common to spring design. The function of the spring in many applications is to provide a modest restoring force as parts undergo large displacement in carrying out kinematic functions. Mathematically, this performance requirement ($\delta \geq \Delta$) is stated in an inequality form, using Eq. (a), as

$$\frac{P}{K} \geq \Delta \tag{g}$$

Shear-stress constraint. To prevent material overstressing, *shear stress* in the wire must be no greater than τ_a, which is expressed in mathematical form as

$$\tau \leq \tau_a \tag{h}$$

Constraint on the frequency of surge waves. We also wish to avoid resonance in dynamic applications by making the *frequency of surge waves* (along the spring) as great as possible. For the present problem, we require the frequency of surge waves for the spring to be at least ω_0 (Hz). The constraint is expressed in mathematical form as

$$\omega \geq \omega_0 \tag{i}$$

Diameter constraint. The *outer diameter* of the spring should not be greater than D_0, so

$$D + d \leq D_0 \tag{j}$$

Explicit bounds on design variables. To avoid fabrication and other practical difficulties, we put *minimum and maximum size limits* on the wire diameter, coil diameter, and number of turns:

$$\begin{aligned} d_{\min} &\leq d \leq d_{\max} \\ D_{\min} &\leq D \leq D_{\max} \\ N_{\min} &\leq N \leq N_{\max} \end{aligned} \tag{k}$$

Mathematical formulation. Thus, the purpose of the minimum-mass spring design problem is to select the design variables d, D, and N to minimize the mass of Eq. (f), while satisfying the ten inequality constraints of Eqs. (g)–(k). If the intermediate variables are eliminated, the problem formulation can be summarized in terms of the design variables only.

EXAMPLE 2.10 FORMULATION OF THE SPRING DESIGN PROBLEM WITH DESIGN VARIABLES ONLY

A summary of the problem formulation for the optimum design of coil springs is as follows:

Specified data: Q, P, ρ, γ, τ_a, G, Δ, ω_0, D_0, $d_{\min}$, $d_{\max}$, $D_{\min}$, $D_{\max}$, $N_{\min}$, $N_{\max}$
Design variables: d, D, N
Cost function: Minimize the mass of the spring given in Eq. (f).

Constraints:
Deflection limit:

$$\frac{8PD^3N}{d^4G} \geq \Delta \tag{l}$$

Shear stress:

$$\frac{8PD}{\pi d^3}\left[\frac{(4D-d)}{4(D-d)}+\frac{0.615d}{D}\right] \leq \tau_a \tag{m}$$

Frequency of surge waves:

$$\frac{d}{2\pi ND^2}\sqrt{\frac{G}{2\rho}} \geq \omega_0 \tag{n}$$

Diameter constraint: Given in Eq. (j).
Design variable bounds: Given in Eq. (k).

Mathematical formulation. Thus the optimization problem is to determine the design variables d, D and N to minimize the cost function in Eq. (f) subject to the constraints in Eq. (j), (k) and (l)–(n). The problem is solved optimum solution using a numerical optimization method in chapter: More on Numerical Methods for Constrained Optimum Design.

2.10 MINIMUM-WEIGHT DESIGN OF A SYMMETRIC THREE-BAR TRUSS

Step 1: *Project/problem description.* As an example of a slightly more complex design problem, consider the three-bar structure shown in Fig. 2.9 (Schmit, 1960; Haug and Arora, 1979). This is a statically indeterminate structure for which the member forces cannot be calculated solely from equilibrium equations. The structure is to be designed for minimum volume (or, equivalently, minimum mass) to support a force P. It must satisfy various performance and technological constraints, such as member crushing, member buckling, failure by excessive deflection of node 4, and failure by resonance when the natural frequency of the structure is below a given threshold.

Step 2: *Data and information collection.* Geometry data, properties of the material used, and loading data are needed to solve the problem. In addition, since the structure is statically indeterminate, the static equilibrium equations alone are not enough to analyze it. We need to use advanced analysis procedures to obtain expressions for member forces, nodal displacements, and the natural frequency to formulate constraints for the problem. Here we will give such expressions.

Since the structure must be symmetric, members 1 and 3 will have the same cross-sectional area, say A_1. Let A_2 be the cross-sectional area of member 2. Using analysis procedures for

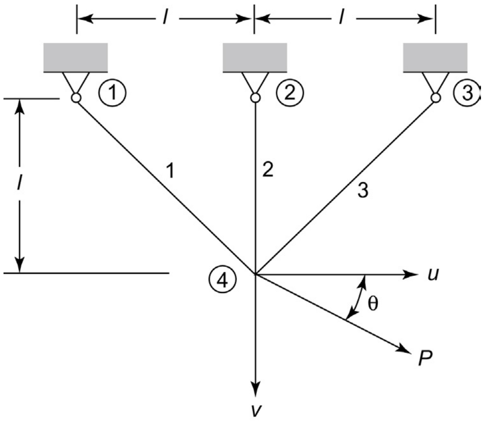

FIGURE 2.9 **Three-bar truss.**

statically indeterminate structures, horizontal and vertical displacements u and v of node 4 are calculated as

$$u = \frac{\sqrt{2}lP_u}{A_1E}; \quad v = \frac{\sqrt{2}lP_v}{(A_1 + \sqrt{2}A_2)E} \tag{a}$$

where E is the modulus of elasticity for the material, P_u and P_v are the horizontal and vertical components of the applied load P given as $P_u = P\cos\theta$ and $P_v = P\sin\theta$, and l is the height of the truss as shown in Fig. 2.9. Using the displacements, forces carried by the members of the truss can be calculated. Then the stresses σ_1, σ_2, and σ_3 in members 1, 2, and 3 under the applied load P can be computed from member forces as (stress = force/area; $\sigma_i = F_i/A_i$):

$$\sigma_1 = \frac{1}{\sqrt{2}}\left[\frac{P_u}{A_1} + \frac{P_v}{A_1 + \sqrt{2}A_2}\right] \tag{b}$$

$$\sigma_2 = \frac{\sqrt{2}P_v}{(A_1 + \sqrt{2}A_2)} \tag{c}$$

$$\sigma_3 = \frac{1}{\sqrt{2}}\left[-\frac{P_u}{A_1} + \frac{P_v}{A_1 + \sqrt{2}A_2}\right] \tag{d}$$

Note that the member forces, and hence stresses, are dependent on the design of the structure, that is, the member areas.

Many structures support moving machinery and other dynamic loads. These structures vibrate with a certain frequency known as *natural frequency*. This is an intrinsic dynamic property of a structural system. There can be several modes of vibration, each having its own frequency. *Resonance* causes catastrophic failure of the structure, which occurs when any one of its vibration frequencies coincides with the frequency of the operating machinery it supports, or frequency of the applied loads.

Therefore, it is reasonable to demand that no structural frequency be close to the frequency of the operating machinery. The mode of vibration corresponding to the lowest natural frequency is important because that mode is excited first. It is important to make the lowest (fundamental) natural frequency of the structure as high as possible to avoid any possibility of resonance. This also makes the structure stiffer. Frequencies of a structure are obtained by solving an eigenvalue problem involving the structure's stiffness and mass properties. The lowest eigenvalue ζ related to the lowest natural frequency of the symmetric three-bar truss is computed using a consistent-mass model:

$$\varsigma = \frac{3EA_1}{\rho l^2 (4A_1 + \sqrt{2}A_2)} \tag{e}$$

where ρ is the material mass per unit volume (mass density). This completes the analysis of the structure.

Step 3: *Definition of design variables.* The following design variables are defined for the symmetric three-bar truss:

A_1 = cross-sectional area of material for members 1 and 3
A_2 = cross-sectional area of material for member 2

Other design variables for the problem are possible depending on the cross-sectional shape of members, as shown in Fig. 2.6.

Step 4: *Optimization criterion.* The relative merit of any design for the problem is measured in its material weight. Therefore, the total weight of three members of the truss serves as a cost function (weight of a member = cross-sectional area × length × weight density):

$$Volume = l\gamma(2\sqrt{2}A_1 + A_2) \tag{f}$$

where γ is the weight density and l is the height of the truss.

Step 5: *Formulation of constraints.* The truss structure is designed for use in two applications. In each application, it supports different loads. These are called loading conditions for the structure. In the present application, a symmetric structure is obtained if the following two loading conditions are considered. The first load is applied at an angle θ and the second one, of same magnitude, at an angle $(\pi-\theta)$, where the angle θ ($0° \le \theta \le 90°$) is shown earlier in Fig. 2.9. If we let member 1 be the same as member 3, then the second loading condition can be ignored. Since we are designing a symmetric structure, we consider only one load applied at an angle θ ($0° \le \theta \le 90°$).

Note from Eqs. (b) and (c) that the stresses σ_1 and σ_2 are always positive (tensile). If $\sigma_a > 0$ is an allowable stress for the material, then the *stress constraints* for members 1 and 2 are

$$\sigma_1 \le \sigma_a; \sigma_2 \le \sigma_a \tag{g}$$

However, from Eq. (c), stress in member 3 can be positive (tensile) or negative (compressive) depending on the load angle. Therefore, both possibilities need to be considered in formulating the stress constraint for member 3. One way to formulate such a constraint was explained in Section 2.5 for the two-bar truss. Another way is as follows:

$$\text{IF } (\sigma_3 < 0) \text{ THEN } -\sigma_3 \le \sigma_a \text{ ELSE } \sigma_3 \le \sigma_a \tag{h}$$

Since the sign of the stress does not change with design, if the member is in compression, it remains in compression throughout the optimization process. Therefore, the constraint function remains continuous and differentiable.

A similar procedure can be used for stresses in bars 1 and 2 if the stresses can reverse their sign (eg, when the load direction is reversed). Horizontal and vertical deflections of node 4 must be within the specified limits Δ_u and Δ_v, respectively. Using Eq. (a), the *deflection constraints* are

$$u \le \Delta_u ; \upsilon \le \Delta_\upsilon \tag{i}$$

As discussed previously, the *fundamental natural frequency* of the structure should be higher than a specified frequency ω_0 (Hz). This constraint can be written in terms of the lowest eigenvalue for the structure. The eigenvalue corresponding to a frequency of ω_0 (Hz) is given as $(2\pi\omega_0)^2$. The lowest eigenvalue ζ for the structure given in Eq. (e) should be higher than $(2\pi\omega_0)^2$, that is,

$$\varsigma \ge (2\pi\omega_0)^2 \tag{j}$$

To impose *buckling constraints* for members under compression, an expression for the moment of inertia of the cross-section is needed. This expression cannot be obtained because the cross-sectional shape and dimensions are not specified. However, the moment of inertia I can be related to the cross-sectional area of the members as $I = \beta A^2$, where A is the cross-sectional area and β is a nondimensional constant. This relation follows if the shape of the cross-section is fixed and all of its dimensions are varied in the same proportion.

The axial force for the ith member is given as $F_i = A_i\sigma_i$, where $i = 1, 2, 3$ with tensile force taken as positive. Members of the truss are considered columns with pin ends. Therefore, the buckling load for the ith member is given as $\pi^2 EI/l_i^2$, where l_i is the length of the ith member (Crandall et al., 2012). Buckling constraints are expressed as $-F_i \le \pi^2 EI/l_i^2$, where $i = 1, 2, 3$. The negative sign for F_i is used to make the left side of the constraints positive when the member is in compression. Also, there is no need to impose buckling constraints for members in tension. With the foregoing formulation, the buckling constraint for tensile members is automatically satisfied. Substituting various quantities, buckling constraints for three members of the truss are

$$-\sigma_1 \le \frac{\pi^2 E\beta A_1}{2l^2} \le \sigma_a ; \; -\sigma_2 \le \frac{\pi^2 E\beta A_2}{l^2} \le \sigma_a ; \; -\sigma_3 \le \frac{\pi^2 E\beta A_1}{2l^2} \le \sigma_a \tag{k}$$

Note that the buckling load has been divided by the member area to obtain the buckling stress in Eq. (k). The buckling stress is required not to exceed an allowable buckling stress σ_a. It is additionally noted that with the foregoing formulation, the load P in Fig. 2.9 can be applied in the positive or negative direction. When the load is applied in the opposite direction, the member forces are also reversed. The foregoing formulation for the buckling constraints can treat both positive and negative load in the solution process.

Finally, A_1 and A_2 must both be non-negative, that is, A_1, $A_2 \geq 0$. Most practical design problems require each member to have a certain minimum area, A_{min}. Therefore the minimum area constraints are written as

$$A_1, A_2 \geq A_{min} \tag{l}$$

Mathematical formulation. The optimum design problem, then, is to find cross-sectional areas A_1, $A_2 \geq A_{min}$ to minimize the volume of Eq. (f) subject to the constraints of Eqs. (g)–(l). This small-scale problem has 11 inequality constraints and 2 design variables. The problem is solved for optimum solution using a numerical optimization method in chapter: Practical Applications of Optimization.

2.11 A GENERAL MATHEMATICAL MODEL FOR OPTIMUM DESIGN

To describe optimization concepts and methods, we need a general mathematical statement for the optimum design problem. Such a mathematical model is defined as the minimization of a cost function while satisfying all equality and inequality constraints. The inequality constraints in the model are always transformed as "≤ types." This will be called the *standard design optimization model* that is treated throughout this text. In the optimization literature, this model is also called *nonlinear programming problem* (NLP). It will be shown that all design problems can easily be transcribed into this standard form.

2.11.1 Standard Design Optimization Model

In previous sections, several design problems were formulated. All problems have an optimization criterion that can be used to compare various designs and to determine an optimum or the best one. Most design problems must also satisfy performance constraints and other limitations. Some design problems have only inequality constraints, others have only equality constraints, and some have both. We can define a general mathematical model for optimum design to encompass all of these possibilities. A standard form of the model is first stated, and then transformation of various problems into the standard form is explained.

Standard Design Optimization Model

Find an n-vector $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ of design variables to

Minimize a cost function:

$$f(\mathbf{x}) = f(x_1, x_2 \ldots, x_n) \tag{2.1}$$

subject to the p equality constraints:

$$h_j(\mathbf{x}) = h_j(x_1, x_2, \ldots, x_n) = 0;\ j = 1 \text{ to } p \tag{2.2}$$

and the m inequality constraints:

$$g_i(\mathbf{x}) = g_i(x_1, \ldots, x_n) \leq 0;\ i = 1 \text{ to } m \tag{2.3}$$

Note that the simple bounds on design variables, such as $x_i \geq 0$, or $x_{iL} \leq x_i \leq x_{iU}$, where x_{iL} and x_{iU} are the smallest and largest allowed values for x_i, are assumed to be included in the inequalities of Eq. (2.3). In numerical methods, these constraints are treated explicitly to take advantage of their simple form to achieve efficiency. However, in discussing the basic optimization concepts, we assume that the inequalities in Eq. (2.3) include these constraints as well.

2.11.2 Maximization Problem Treatment

The general design model treats only minimization problems. This is no restriction, as maximization of a function $F(\mathbf{x})$ is the same as minimization of a transformed function $f(\mathbf{x}) = -F(\mathbf{x})$. To see this graphically, consider a plot of the function of one variable $F(x)$, shown in Fig. 2.10a. The function $F(x)$ takes its maximum value at the point x^*. Next consider a graph of the function $f(x) = -F(x)$, shown in Fig. 2.10b. It is seen that $f(x)$ is a reflection of $F(x)$ about the x-axis. It is also seen from the graph that $f(x)$ takes on a minimum value at the same point x^* where the maximum of $F(x)$ occurs. Therefore, minimization of $f(x)$ is equivalent to maximization of $F(x)$.

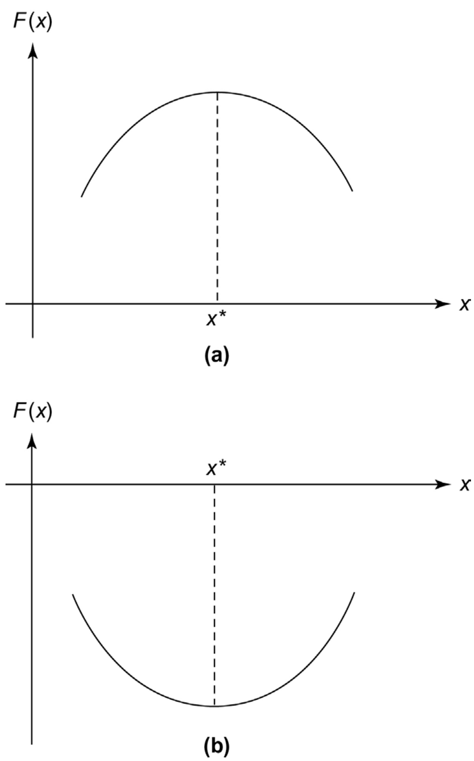

FIGURE 2.10 **Point maximizing $F(x)$ equals point minimizing $-F(x)$.** (a) Plot of $F(x)$; (b) plot of $f(x) = -F(x)$.

2.11.3 Treatment of "Greater Than Type" Constraints

The standard design optimization model treats only "≤ type" inequality constraints. Many design problems may also have "≥ type" inequalities. Such constraints can be converted to the standard form without much difficulty. The "≥ type" constraint $G_j(\mathbf{x}) \geq 0$ is equivalent to the "≤ type" inequality $g_j(\mathbf{x}) = -G_j(\mathbf{x}) \leq 0$. Therefore, we can multiply any "≥ type" constraint by −1 to convert it to a "≤ type."

2.11.4 Application to Different Engineering Fields

Design optimization problems from different fields of engineering can be transcribed into the standard model. However, *the overall process of designing different engineering systems is the same*. Analytical and numerical methods for analyzing systems can differ. Formulation of the design problem can contain terminology that is specific to the particular domain of application. For example, in the fields of structural, mechanical, and aerospace engineering, we are concerned with the integrity of the structure and its components. The performance requirements involve constraints on member stresses, strains, deflections at key points, frequencies of vibration, buckling failure, and so on. Such concepts are specific to each field, and designers working in the particular field understand their meaning and the constraints.

Other fields of engineering also have their own terminology to describe design optimization problems. However, once the problems from different fields have been transcribed into mathematical statements using a standard notation, they have the same mathematical form. They are contained in the standard design optimization model defined in Eqs. (2.1) to (2.3). For example, all of the problems formulated earlier in this chapter can be transformed into the form of Eqs. (2.1) to (2.3). The optimization concepts and methods described in the text are quite general and can be used to solve problems from diverse fields. *The methods can be developed without reference to any design application.* This is a key point that must be kept in mind while studying the optimization concepts and methods.

2.11.5 Important Observations about the Standard Model

Several features of the standard model must be clearly understood:

1. *Dependence of functions on design variables*: First of all, the functions $f(\mathbf{x})$, $h_j(\mathbf{x})$, and $g_i(\mathbf{x})$ must *depend*, explicitly or implicitly, on some of the *design variables*. Only then are they valid for the design problem. Functions that do not depend on any variable have no relation to the problem and can be safely ignored.
2. *Number of equality constraints*: The number of *independent equality constraints* must be less than, or at the most equal to, the number of design variables (ie, $p \leq n$). When $p > n$, we have an *overdetermined system* of equations. In that case, either some *equality constraints* are *redundant* (linearly dependent on other constraints) or they are *inconsistent*. In the former case, redundant constraints can be deleted and, if $p < n$, the optimum solution for the problem is possible. In the latter case, no solution for the design problem is possible and the problem formulation needs to be closely reexamined. When $p = n$, no optimization of the system is necessary because the roots of the equality constraints are the only candidate points for optimum design.

3. *Number of inequality constraints*: Although there is a limitation on the number of independent equality constraints, *there is no restriction on the number of inequality constraints*. However, the total number of active constraints (satisfied at equality) must, at the optimum, be less than or at the most equal to the number of design variables.
4. *Unconstrained problems*: Some design problems may not have any constraints. These are called *unconstrained*; those with constraints are called *constrained*.
5. *Linear programming problems*: If all of the functions $f(\mathbf{x})$, $h_j(\mathbf{x})$, and $g_i(\mathbf{x})$ are linear in design variables **x**, then the problem is called a *linear programming problem*. If any of these functions is nonlinear, the problem is called a *nonlinear programming problem*.
6. *Scaling of problem functions*: It is important to note that if the *cost function is scaled* by multiplying it with a positive constant, the optimum design does not change. However, the optimum cost function value does change. Also, any constant can be added to the cost function without affecting the optimum design. Similarly, the inequality *constraints can be scaled* by any positive constant and the equalities by any constant. This will not affect the feasible region and hence the optimum solution. All the foregoing transformations, however, affect the values of the *Lagrange multipliers* (defined in chapter: Optimum Design Concepts: Optimality Conditions). Also, performance of the numerical algorithms for a solution to the optimization problem may be affected by these transformations.

2.11.6 Feasible Set

The term *feasible set* will be used throughout the text. *A feasible set for the design problem is a collection of all feasible designs.* The terms *constraint set* and *feasible design space* are also used to represent the feasible set of designs. The letter S is used to represent the feasible set. Mathematically, the set S is a collection of design points satisfying all constraints:

$$S = (\mathbf{x} \mid h_j(\mathbf{x}) = 0,\ j = 1 \text{ to } p;\ g_i(\mathbf{x}) \le 0,\ i = 1 \text{ to } m) \tag{2.4}$$

The *set of feasible designs* is sometimes referred to as the *feasible region*, especially for optimization problems with two design variables. It is important to note that the *feasible region usually shrinks when more constraints are added to the design model and expands when some constraints are deleted*. When the feasible region shrinks, the number of possible designs that can optimize the cost function is reduced; that is, there are fewer feasible designs. In this event, the minimum value of the cost function is likely to increase. The effect is completely opposite when some constraints are dropped. This observation is significant for practical design problems and should be clearly understood.

2.11.7 Active/Inactive/Violated Constraints

We will quite frequently refer to a constraint as *active*, *tight*, *inactive*, or *violated*. We define these terms precisely. An inequality constraint $g_j(\mathbf{x}) \le 0$ is said to be *active* at a design point $\mathbf{x}^*$ if it is satisfied at equality (ie, $g_j(\mathbf{x}^*) = 0$). This is also called a *tight* or *binding* constraint. For a feasible design, an inequality constraint may or may not be active. However, all equality constraints are active for all feasible designs.

An inequality constraint $g_j(\mathbf{x}) \leq 0$ is said to be *inactive* at a design point $\mathbf{x}^*$ if it is strictly satisfied (ie, $g_j(\mathbf{x}^*) < 0$). It is said to be *violated* at a design point $\mathbf{x}^*$ if its value is positive (ie, $g_j(\mathbf{x}^*) > 0$). An *equality constraint* $h_i(\mathrm{x}) = 0$ is violated at a design point $\mathbf{x}^*$ if $h_i(\mathbf{x}^*)$ is not identically zero. Note that by these definitions, an equality constraint is either active or violated at a given design point.

2.11.8 Discrete and Integer Design Variables

So far, we have assumed in the standard model that variables x_i can have any numerical value within the feasible region. Many times, however, some variables are required to have discrete or integer values. Such variables appear quite often in engineering design problems. We encountered problems in Sections 2.4, 2.6, and 2.9 that have integer design variables. Before describing how to treat them, let us define what we mean by discrete and integer variables.

A design variable is called *discrete* if its value must be selected from a given finite set of values. For example, a plate thickness must be the one that is available commercially: 1/8, 1/4, 3/8, 1/2, 5/8, 3/4, 1 in, and so on. Similarly, structural members must be selected from a catalog to reduce fabrication cost. Such variables must be treated as discrete in the standard formulation.

An *integer variable*, as the name implies, must have an integer value; for example, the number of logs to be shipped, the number of bolts used, the number of coils in a spring, the number of items to be shipped, and so on. Problems with such variables are called *discrete* and *integer programming problems*. Depending on the type of problem functions, the problems can be classified into five different categories. These classifications and the methods to solve them are discussed in chapter: Discrete Variable Optimum Design Concepts and Methods.

In some sense, discrete and integer variables impose additional constraints on the design problem. Therefore, as noted before, the optimum value of the cost function is likely to increase with these variables compared with the same problem that is solved with continuous variables. If we treat all design variables as continuous, the minimum value of the cost function represents a lower bound on the true minimum value when discrete or integer variables are used. This gives some idea of the "best" optimum solution if all design variables are treated as continuous. The optimum cost function value is likely to increase when discrete values are assigned to variables. Thus, the first suggested procedure is to solve the problem assuming continuous design variables if possible. Then the nearest discrete/integer values are assigned to the variables and the design is checked for feasibility. With a few trials, the best feasible design close to the continuous optimum can be obtained.

As a second approach for solving such problems, an *adaptive numerical optimization procedure* may be used. An optimum solution with continuous variables is first obtained if possible. Then only the variables that are close to their discrete or integer value are assigned that value. They are held fixed and the problem is optimized again. The procedure is continued until all variables have been assigned discrete or integer values. A few further trials may be carried out to improve the optimum cost function value. This procedure has been demonstrated by Arora and Tseng (1988).

The foregoing procedures require additional computational effort and do not guarantee a true minimum solution. However, they are quite straightforward and do not require any additional methods or software for solution of discrete/integer variable problems.

2.11.9 Types of Optimization Problems

The standard design optimization model can represent many different problem types. We saw that it can be used to represent linear programming, and unconstrained and constrained, nonlinear programming optimization problems. It is important to understand other optimization problems that are encountered in practical applications. Many times these problems can be transformed into the standard model and solved by the optimization methods presented and discussed in this text. Here we present an overview of the types of optimization problems.

Continuous/Discrete-Variable Optimization Problems

When the design variables can have any numerical value within their allowable range, the problem is called a *continuous-variable* optimization problem. When the problem has only discrete/integer variables, it is called a *discrete/integer-variable* optimization problem. When the problem has both continuous and discrete variables, it is called a mixed-variable optimization problem. Numerical methods for these types of problems have been developed, as discussed in later chapters.

Smooth/Nonsmooth Optimization Problems

When its functions are continuous and differentiable, the problem is referred to as smooth (*differentiable*). There are numerous practical optimization problems in which the functions can be formulated as continuous and differentiable. There are also many practical applications where the problem functions are not differentiable or even discontinuous. Such problems are called nonsmooth (*nondifferentiable*).

Numerical methods to solve these two classes of problems can be different. Theory and numerical methods for smooth problems are well developed. Therefore, it is most desirable to formulate the problem with continuous and differentiable functions as far as possible. Sometimes, a problem with discontinuous or nondifferentiable functions can be transformed into one that has continuous and differentiable functions so that optimization methods for smooth problems can be used. Such applications are discussed in chapter: Practical Applications of Optimization.

Problems with Implicit Constraints

Some constraints are quite simple, such as the smallest and largest allowable values for the design variables, whereas more complex ones may be indirectly influenced by the design variables. For example, deflection at a point in a large structure depends on its design. However, it is impossible to express deflection as an explicit function of the design variables except for very simple structures. These are called *implicit constraints*. When there are implicit functions in the problem formulation, it is not possible to formulate the problem functions explicitly in terms of design variables alone. Instead, we must use some *intermediate variables* in the problem formulation. We will discuss formulations having implicit functions in chapter: Practical Applications of Optimization.

Network Optimization Problems

A network or a graph consists of points and lines connecting pairs of points. Network models are used to represent many practical problems and processes from different branches of engineering, computer science, operations research, transportation, telecommunication, decision support, manufacturing, airline scheduling, and many other disciplines. Depending on the application type, network optimization problems have been classified as transportation problems, assignment problems, shortest-path problems, maximum-flow problems, minimum-cost-flow problems, and critical path problems.

To understand the concept of network problems, let us describe the transportation problem in more detail. Transportation models play an important role in logistics and supply chain management for reducing cost and improving service. Therefore the goal is to find the most effective way to transport goods. A shipper having m warehouses with supply s_i of goods at the ith warehouse must ship goods to n geographically dispersed retail centers, each with a customer demand d_j that must be met. The objective is to determine the minimum cost distribution system, given that the unit cost of transportation between the ith warehouse and the jth retail center is c_{ij}.

This problem can be formulated as one of linear programming. Since such network optimization problems are encountered in diverse fields, special methods have been developed to solve them more efficiently and perhaps in real time. Many textbooks are available on this subject. We do not address these problems in any detail, although some of the methods presented in Chapters 15–19 can be used to solve them.

Dynamic-Response Optimization Problems

Many practical systems are subjected to transient dynamic loads. In such cases, some of the problem constraints are time-dependent. Each of these constraints must be imposed for the entire time interval of interest. Therefore each represents an infinite set of constraints because the constraint must be imposed at each time point in the given interval. The usual approach to treating such a constraint is to impose it at a finite number of time points in the given interval. This way the problem is transformed into the standard form and treated with the methods presented in this textbook.

Design Variables as Functions

In some applications, the design variables are not parameters but functions of one, two, or even three variables. Such design variables arise in optimal control problems where the input needs to be determined over the desired range of time to control the behavior of the system. The usual treatment of design functions is to parameterize them. In other words, each function is represented in terms of some known functions, called the *basis functions*, and the parameters multiplying them. The parameters are then treated as design variables. In this way the problem is transformed into the standard form and the methods presented in this textbook can be used to solve it.

2.12 DEVELOPMENT OF PROBLEM FORMULATION FOR PRACTICAL APPLICATIONS

On the basis of experience, it is noted that usually several iterations are needed before an acceptable formulation for a practical design optimization problem is obtained. In any case, one has to start with an initial formulation for the problem. When a solution is sought for this initial formulation, several flaws may be detected that need to be rectified by trial and error iterative process. For example, the solution algorithm may not be able to satisfy all the constraints; that is, there is no feasible design for the problem. In this case, one needs to determine the offending constraints and redefine them so that there are feasible designs for the problem. This in itself may require several iterations.

In other cases, the solution process can find feasible designs for the problem but it cannot converge to an optimum solution. In such cases, the feasible set is most likely unbounded and realistic bounds need to be defined for the design variables of the problem. In yet other cases, the solution process can converge to an optimum solution but the solution is weird and impractical. In such cases, perhaps some practical performance requirements have not been included in the formulation; or, practical bounds may need to be defined for the design variables of the problem and the problem needs to be solved again.

In some cases, the entire formulation for the problem may need to be re-examined if the solution process does not yield an optimum solution, or it gives an unrealistic solution. In these cases, the design variables, the optimization criterion, and all the constraints may need to be re-examined and re-formulated. Sometimes, additional objective functions may need to be introduced into the formulation to obtain practical solutions. With more than one objective function in the formulation, multiobjective optimization methods will need to be used to solve the problem.

Thus we see that several modifications of the initial formulation may be needed in an iterative manner before a proper formulation for a practical problem is achieved. Each modification requires the problem to be solved using an efficient numerical optimization algorithm and the associated software. Further discussion of this important topic is presented in chapter: Optimum Design: Numerical Solution Process and Excel Solver.

Development of a proper formulation for optimization of a practical design problem is an iterative process requiring several trial runs before an acceptable formulation is realized.

EXERCISES FOR CHAPTER 2

Transcribe the problem statements to mathematical formulation for optimum design

2.1 A 100×100-m lot is available to construct a multistory office building. At least 20,000 m^2 of total floor space is needed. According to a zoning ordinance, the maximum height of the building can be only 21 m, and the parking area outside the building must be at least 25% of the total floor area. It has been decided to fix the height of each story at 3.5 m. The cost of the building in millions of dollars is estimated at $0.6h + 0.001A$, where A is the cross-sectional area of the building per floor and h is the height of the building. Formulate the minimum-cost design problem.

2.2 A refinery has two crude oils:

1. Crude A costs $120/barrel (bbl) and 20,000 bbl are available.
2. Crude B costs $150/bbl and 30,000 bbl are available.

The company manufactures gasoline and lube oil from its crudes. Yield and sale price per barrel and markets are shown in Table E2.2. How much crude oil should the company use to maximize its profit? Formulate the optimum design problem.

TABLE E2.2 Data for Refinery Operations

Product	Yield/bbl		Sale price per bbl ($)	Market (bbl)
	Crude A	Crude B		
Gasoline	0.6	0.8	200	20,000
Lube oil	0.4	0.2	400	10,000

2.3 Design a beer mug, shown in Fig. E2.3, to hold as much beer as possible. The height and radius of the mug should be no more than 20 cm. The mug must be at least 5 cm in radius. The surface area of the sides must be no greater than 900 cm^2 (ignore the bottom area of the mug and mug handle). Formulate the optimum design problem.

2.4 A company is redesigning its parallel-flow heat exchanger of length l to increase its heat transfer. An end view of the unit is shown in Fig. E2.4. There are certain limitations on the design problem. The smallest available conducting tube has a radius of 0.5 cm, and all tubes must be of the same size. Further, the total cross-sectional area of all of the tubes cannot exceed 2000 cm^2 to ensure adequate space inside the outer shell. Formulate the problem to determine the number of tubes and the radius of each one to maximize the surface area of the tubes in the exchanger.

2.5 Proposals for a parking ramp have been defeated, so we plan to build a parking lot in the downtown urban renewal section. The cost of land is $200W + 100D$, where W is the width along the street and D is the depth of the lot in meters. The available width along the street is 100 m, whereas the maximum depth available is 200 m. We want the size of the lot to be at least 10,000 m^2. To avoid unsightliness, the city requires that the

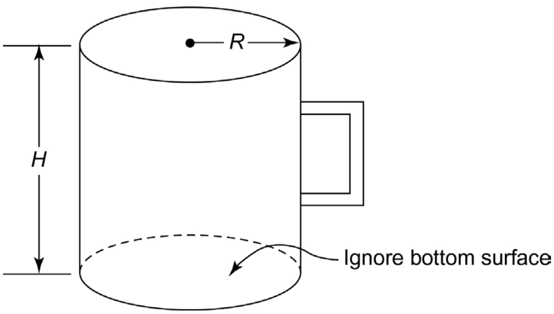

FIGURE E2.3 **Beer mug.**

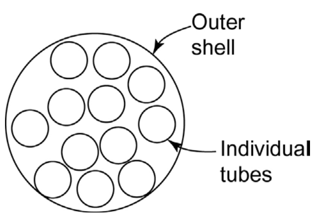

FIGURE E2.4 **Cross-section of a heat exchanger.**

TABLE E2.7 Data for the Diet Problem

Vitamin	1 kg bread provides	1 kg milk provides
A	1 unit	2 units
B	3 units	2 units
Cost/kg, $	2	1

longer dimension of any lot be no more than twice the shorter dimension. Formulate the minimum-cost design problem.

2.6 A manufacturer sells products A and B. Profit from A is $10/kg and is $8/kg from B. Available raw materials for the products are 100 kg of C and 80 kg of D. To produce 1 kg of A, we need 0.4 kg of C and 0.6 kg of D. To produce 1 kg of B, we need 0.5 kg of C and 0.5 kg of D. The markets for the products are 70 kg for A and 110 kg for B. How much of A and B should be produced to maximize profit? Formulate the design optimization problem.

2.7 Design a diet of bread and milk to get at least 5 units of vitamin A and 4 units of vitamin B daily. The amount of vitamins A and B in 1 kg of each food and the cost per kilogram of the food are given in Table E2.7. For example, one kg of bread costs 2$ and provides one unit of vitamin A and 3 units of vitamin B. Formulate the design optimization problem so that we get at least the basic requirements of vitamins at the minimum cost.

2.8 Enterprising engineering students have set up a still in a bathtub. They can produce 225 bottles of pure alcohol each week. They bottle two products from alcohol: (1) wine, at 20 proof, and (2) whiskey, at 80 proof. Recall that pure alcohol is 200 proof. They have an unlimited supply of water, but can only obtain 800 empty bottles per week because of stiff competition. The weekly supply of sugar is enough for either 600 bottles of wine or 1200 bottles of whiskey. They make a $1.00 profit on each bottle of wine and a $2.00 profit on each bottle of whiskey. They can sell whatever they produce. How many bottles of wine and whiskey should they produce each week to maximize profit? Formulate the design optimization problem (created by D. Levy).

2.9 Design a can closed at one end using the smallest area of sheet metal for a specified interior volume of 600 m^3. The can is a right-circular cylinder with interior height h and radius r. The ratio of height to diameter must not be less than 1.0 nor greater than 1.5. The height cannot be more than 20 cm. Formulate the design optimization problem.

2.10 Design a shipping container closed at both ends with dimensions $b \times b \times h$ to minimize the ratio: (round-trip cost of shipping container only)/(one-way cost of shipping contents only). Use the data in Table E2.10. Formulate the design optimization problem.

TABLE E2.10 Data for Shipping Container

Mass of container/surface area	80 kg/m^2
Maximum b	10 m
Maximum h	18 m
One-way shipping cost, full or empty	$18/kg gross mass
Mass of contents	150 kg/m^3

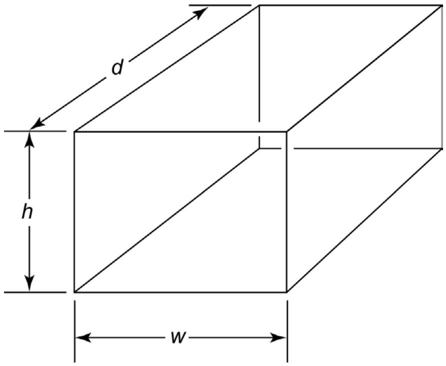

FIGURE E2.13 **Steel frame.**

2.11 Certain mining operations require an open-top rectangular container to transport materials. The data for the problem are as follows:
Construction costs:
- *Sides*: \$50/m²
- *Ends*: \$60/m²
- *Bottom*: \$90/m²

Minimum volume needed: 150 m³
Formulate the problem of determining the container dimensions at a minimum cost.

2.12 Design a circular tank closed at both ends to have a volume of 250 m³. The fabrication cost is proportional to the surface area of the sheet metal and is \$400/m². The tank is to be housed in a shed with a sloping roof. Therefore, height H of the tank is limited by the relation $H \le (10 - D/2)$, where D is the tank's diameter. Formulate the minimum-cost design problem.

2.13 Design the steel framework shown in Fig. E2.13 at a minimum cost. The cost of a horizontal member in one direction is \$20 w and in the other direction it is \$30 d. The cost of a vertical column is \$50 h. The frame must enclose a total volume of at least 600 m³. Formulate the design optimization problem.

2.14 Two electric generators are interconnected to provide total power to meet the load. Each generator's cost is a function of the power output, as shown in Fig. E2.14. All costs and power are expressed on a per-unit basis. The total power needed is at least 60 units. Formulate a minimum-cost design problem to determine the power outputs P_1 and P_2.

2.15 *Transportation problem*. A company has m manufacturing facilities. The facility at the ith location has capacity to produce b_i units of an item. The product should be shipped to n distribution centers. The distribution center at the jth location requires at least a_j units of the item to satisfy demand. The cost of shipping an item from the ith plant to the jth distribution center is c_{ij}. Formulate a minimum-cost transportation system to meet each of the distribution center's demands without exceeding the capacity of any manufacturing facility.

2.16 *Design of a two-bar truss*. Design a symmetric two-bar truss (both members have the same cross-section), as shown in Fig. E2.16, to support a load W. The truss consists of

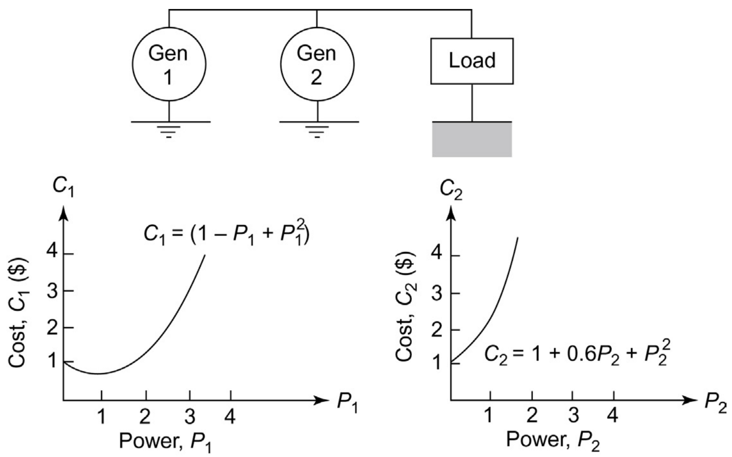

FIGURE E2.14 **Graphic of a power generator.**

two steel tubes pinned together at one end and supported on the ground at the other. The span of the truss is fixed at s. Formulate the minimum-mass truss design problem using height and cross-sectional dimensions as design variables. The design should satisfy the following constraints:

1. Because of space limitations, the height of the truss must not exceed b_1 and must not be less than b_2.
2. The ratio of mean diameter to thickness of the tube must not exceed b_3.
3. The compressive stress in the tubes must not exceed the allowable stress σ_a for steel.
4. The height, diameter, and thickness must be chosen to safeguard against member buckling.

Use the following data: $W = 10$ kN; span $s = 2$ m; $b_1 = 5$ m; $b_2 = 2$ m; $b_3 = 90$; allowable stress $\sigma_a = 250$ MPa; modulus of elasticity $E = 210$ GPa; mass density $\rho = 7850$ kg/m^3; factor of safety against buckling $FS = 2$; $0.1 \le D \le 2$ (m); and $0.01 \le t \le 0.1$ (m).

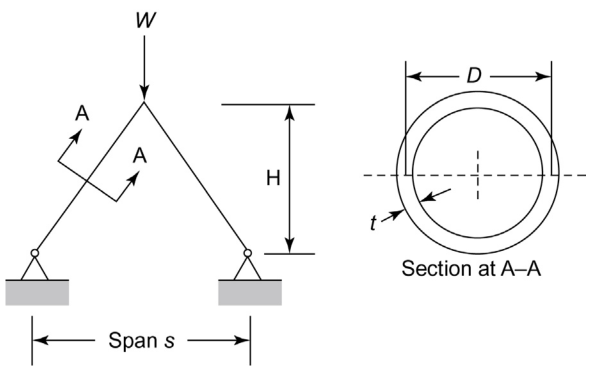

FIGURE E2.16 **Two-bar structure.**

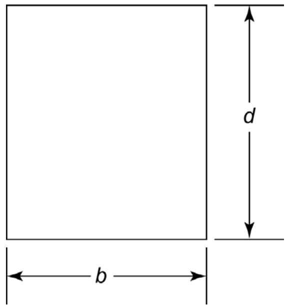

FIGURE E2.17 **Cross-section of a rectangular beam.**

2.17 A beam of rectangular cross-section (Fig. E2.17) is subjected to a maximum bending moment of M and a maximum shear of V. The allowable bending and shearing stresses are σ_a and τ_a, respectively. The bending stress in the beam is calculated as

$$\sigma = \frac{6M}{bd^2}$$

and the average shear stress in the beam is calculated as

$$\tau = \frac{3V}{2bd}$$

where d is the depth and b is the width of the beam. It is also desirable to have the depth of the beam not exceed twice its width. Formulate the design problem for minimum cross-sectional area using this data: $M = 140$ kN m, $V = 24$ kN, $\sigma_a = 165$ MPa, $\tau_a = 50$ MPa.

2.18 A vegetable oil processor wishes to determine how much shortening, salad oil, and margarine to produce to optimize the use its current oil stock supply. At the present time, he has 250,000 kg of soybean oil, 110,000 kg of cottonseed oil, and 2000 kg of milk-base substances. The milk-base substances are required only in the production of margarine. There are certain processing losses associated with each product: 10% for shortening, 5% for salad oil, and no loss for margarine. The producer's back orders require him to produce at least 100,000 kg of shortening, 50,000 kg of salad oil, and 10,000 kg of margarine. In addition, sales forecasts indicate a strong demand for all products in the near future. The profit per kilogram and the base stock required per kilogram of each product are given in Table E2.18. Formulate the problem to maximize profit over the next production-scheduling period (created by J. Liittschwager)

TABLE E2.18 Data for the Vegetable Oil Processing Problem

		Parts per kg of base stock requirements		
Product	**Profit per kg**	**Soybean**	**Cottonseed**	**Milk base**
Shortening	1.00	2	1	0
Salad oil	0.80	0	1	0
Margarine	0.50	3	1	1

Section 2.11: A General Mathematical Model for Optimum Design

2.19 *Answer true or false*:

1. Design of a system implies specification of the design variable values.
2. All design problems have only linear inequality constraints.
3. All design variables should be independent of each other as far as possible.
4. If there is an equality constraint in the design problem, the optimum solution must satisfy it.
5. Each optimization problem must have certain parameters called the design variables.
6. A feasible design may violate equality constraints.
7. A feasible design may violate "≥ type" constraints.
8. A "≤ type" constraint expressed in the standard form is active at a design point if it has zero value there.
9. The constraint set for a design problem consists of all feasible points.
10. The number of independent equality constraints can be larger than the number of design variables for the problem.
11. The number of "≤ type" constraints must be less than the number of design variables for a valid problem formulation.
12. The feasible region for an equality constraint is a subset of that for the same constraint expressed as an inequality.
13. Maximization of $f(x)$ is equivalent to minimization of $1/f(x)$.
14. A lower minimum value for the cost function is obtained if more constraints are added to the problem formulation.
15. Let f_n be the minimum value for the cost function with n design variables for a problem. If the number of design variables for the same problem is increased to, say, $m = 2n$, then $f_m > f_n$, where f_m is the minimum value for the cost function with m design variables.

2.20 A trucking company wants to purchase several new trucks. It has \$2 million to spend. The investment should yield a maximum of trucking capacity for each day in tons × kilometers. Data for the three available truck models are given in Table E2.20: truck load capacity, average speed, crew required per shift, hours of operation for three shifts, and cost of each truck. There are some limitations on the operations that need to be considered. The labor market is such that the company can hire at most 150 truck drivers. Garage and maintenance facilities can handle at the most 25 trucks. How many trucks of each type should the company purchase? Formulate the design optimization problem.

TABLE E2.20 Data for Available Trucks

Truck model	Truck load capacity (tonnes)	Average truck speed (km/h)	Crew required per shift	No. of hours of operations per day (3 shifts)	Cost of each truck ($)
A	10	55	1	18	40,000
B	20	50	2	18	60,000
C	18	50	2	21	70,000

2.21 A large steel corporation has two iron-ore-reduction plants. Each plant processes iron ore into two different ingot stocks, which are shipped to any of three fabricating plants where they are made into either of two finished products. In total, there are two reduction plants, two ingot stocks, three fabricating plants, and two finished products. For the upcoming season, the company wants to minimize total tonnage of iron ore processed in its reduction plants, subject to production and demand constraints. Formulate the design optimization problem and transcribe it into the standard model.

Nomenclature (values for the constants are given in Table E2.21)

$a(r, s)$ = tonnage yield of ingot stock s from 1 ton of iron ore processed at reduction plant r

$b(s, f, p)$ = total yield from 1 ton of ingot stock s shipped to fabricating plant f and manufactured into product p

$c(r)$ = ore-processing capacity in tonnage at reduction plant r

$k(f)$ = capacity of fabricating plant f in tonnage for all stocks

$D(p)$ = tonnage demand requirement for product p

Production and demand constraints:

1. The total tonnage of iron ore processed by both reduction plants must equal the total tonnage processed into ingot stocks for shipment to the fabricating plants.
2. The total tonnage of iron ore processed by each reduction plant cannot exceed its capacity.
3. The total tonnage of ingot stock manufactured into products at each fabricating plant must equal the tonnage of ingot stock shipped to it by the reduction plants.
4. The total tonnage of ingot stock manufactured into products at each fabricating plant cannot exceed the plant's available capacity.
5. The total tonnage of each product must equal its demand.

2.22 *Optimization of a water canal*. Design a water canal having a cross-sectional area of 150 m^2. The lowest construction costs occur when the volume of the excavated material equals the amount of material required for the dykes, that is, $A_1 = A_2$ (see Fig. E2.22). Formulate the problem to minimize the dugout material A_1. Transcribe the problem into the standard design optimization model.

TABLE E2.21 Constants for Iron Ore Processing Operation

$a(1,1) = 0.39$	$c(1) = 1{,}200{,}000$	$k(1) = 190{,}000$	$D(1) = 330{,}000$
$a(1,2) = 0.46$	$c(2) = 1{,}000{,}000$	$k(2) = 240{,}000$	$D(2) = 125{,}000$
$a(2,1) = 0.44$		$k(3) = 290{,}000$	
$a(2,2) = 0.48$			
		$b(1,1,1) = 0.79$	$b(1,1,2) = 0.84$
		$b(2,1,1) = 0.68$	$b(2,1,2) = 0.81$
		$b(1,2,1) = 0.73$	$b(1,2,2) = 0.85$
		$b(2,2,1) = 0.67$	$b(2,2,2) = 0.77$
		$b(1,3,1) = 0.74$	$b(1,3,2) = 0.72$
		$b(2,3,1) = 0.62$	$b(2,3,2) = 0.78$

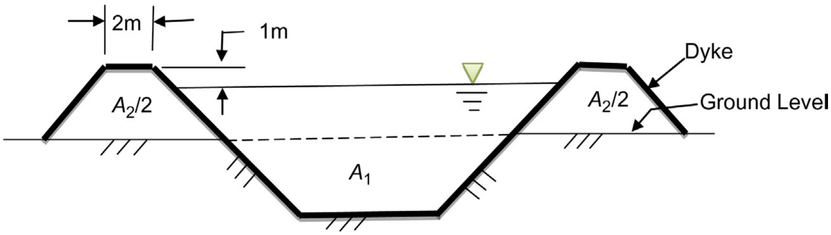

FIGURE E2.22 Cross-section of a canal. *(Created by V. K. Goel.)*

2.23 A cantilever beam is subjected to the point load P (kN), as shown in Fig. E2.23. The maximum bending moment in the beam is PL (kN·m) and the maximum shear is P (kN). Formulate the minimum-mass design problem using a hollow circular cross-section. The material should not fail under bending or shear stress. The maximum bending stress is calculated as

$$\sigma = \frac{PL}{I} R_o \tag{a}$$

where I = moment of inertia of the cross-section. The maximum shearing stress is calculated as

$$\tau = \frac{P}{3I}(R_o^2 + R_o R_i + R_i^2) \tag{b}$$

Transcribe the problem into the standard design optimization model (also use $R_o \le$ 40.0 cm, $R_i \le$ 40.0 cm). Use this data: P = 14 kN; L = 10 m; mass density ρ = 7850 kg/m^3; allowable bending stress σ_b = 165 MPa; allowable shear stress τ_a = 50 MPa.

2.24 Design a hollow circular beam-column, shown in Fig. E2.24, for two conditions: When the axial tensile load P = 50 (kN), the axial stress σ must not exceed an allowable value σ_a, and when P = 0, deflection δ due to self-weight should satisfy the limit $\delta \le 0.001L$. The limits for dimensions are: thickness t = 0.10–1.0 cm, mean radius R = 2.0–20.0 cm, and $R/t \le 20$ (AISC, 2011). Formulate the minimum-weight design problem and transcribe it into the standard form. Use the following data: deflection $\delta = 5wL^4/384EI$; w = self-weight force/length (N/m); σ_a = 250 MPa; modulus of elasticity E = 210 GPa;

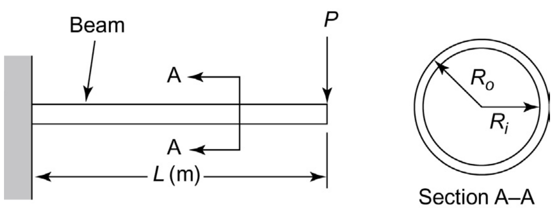

FIGURE E2.23 Cantilever beam.

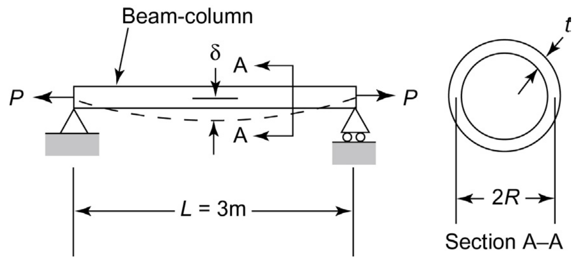

FIGURE E2.24 **Beam column with hollow circular cross-section.**

mass density of beam material $\rho = 7800\ \text{kg/m}^3$; axial stress under load P, $\sigma = P/A$; gravitational constant $g = 9.80\ \text{m/s}^2$; cross-sectional area $A = 2\pi Rt$ (m^2); moment of inertia of beam cross-section $I = \pi R^3 t$ (m^4). Use Newton (N) and millimeters (mm) as units in the formulation.

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