CHAPTER

5

More on Optimum Design Concepts: Optimality Conditions

Upon completion of this chapter, you will be able to:

- Write and use an alternate form of optimality conditions for constrained problems
- Determine if the candidate minimum points are irregular
- Check the second-order optimality conditions at the candidate minimum points for general constrained problems
- Describe duality theory of nonlinear programming

In this chapter, we discuss some additional topics related to the optimality condition for constrained problems. Implications of the regularity requirement in the Karush–Kuhn–Tucker (KKT) necessary conditions are discussed. Second-order optimality conditions for the problem are presented and discussed. These topics are usually not covered in a first course on optimization. Also, they may be omitted in a first reading of this book. They are more suitable for a second course or a graduate level course on the subject.

5.1 ALTERNATE FORM OF KKT NECESSARY CONDITIONS

There is an alternate but entirely equivalent form for the KKT necessary conditions. In this form, the slack variables are not added to the inequality constraints and the conditions of Eqs. (4.46)–(4.52) are written without them. It can be seen that in the necessary conditions of Eqs. (4.46)–(4.52), the slack variable s_i appears in only two equations: Eq. (4.49) as $g_i(\mathbf{x}^*) + s_i^2 = 0$, and Eq. (4.51) as $u_i^* s_i = 0$. We will show that both the equations can be written in an equivalent form without the slack variable s_i.

Consider first Eq. (4.49): $g_i(\mathbf{x}^*) + s_i^2 = 0$ for $i = 1$ to m. The purpose of this equation is to ensure that all the inequalities remain satisfied at the candidate minimum point. The equation can be written as $s_i^2 = -g_i(\mathbf{x}^*)$ and, since $s_i^2 \ge 0$ ensures satisfaction of the constraint, we

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TABLE 5.1 Alternate Form of KKT Necessary Conditions

Problem: Minimize $f(\mathbf{x})$ subject to $h_i(\mathbf{x}) = 0$, $i = 1$ to p; $g_j(\mathbf{x}) \le 0$, $j = 1$ to m

1. *Lagrangian function definition*: $L = f + \sum_{i=1}^{p} v_i h_i + \sum_{j=1}^{m} u_j g_j$ (5.1)

2. *Gradient conditions*: $\frac{\partial L}{\partial x_k} = 0; \quad \frac{\partial f}{\partial x_k} + \sum_{i=1}^{p} v_i^* \frac{\partial h_i}{\partial x_k} + \sum_{j=1}^{m} u_j^* \frac{\partial g_j}{\partial x_k} = 0; \quad k = 1 \text{ to } n$ (5.2)

3. *Feasibility check*: $h_i(\mathbf{x}^*) = 0; \quad i = 1 \text{ to } p; \quad g_j(\mathbf{x}^*) \le 0; \quad j = 1 \text{ to } m$ (5.3)

4. *Switching conditions*: $u_j^* g_j(\mathbf{x}^*) = 0; \quad j = 1 \text{ to } m$ (5.4)

5. *Nonnegativity of Lagrange multipliers for inequalities*: $u_j^* \ge 0; \quad j = 1 \text{ to } m$ (5.5)

6. *Regularity check*: Gradients of active constraints must be linearly independent. In such a case, the Lagrange multipliers for the constraints are unique.

get $-g_i(\mathbf{x}^*) \ge 0$, or $g_i(\mathbf{x}^*) \le 0$ for $i = 1$ to m. Thus, Eq. (4.49), $g_i(\mathbf{x}^*) + s_i^2 = 0$ along with $s_i^2 \ge 0$, can be simply replaced by $g_i(\mathbf{x}^*) \le 0$.

The second equation involving the slack variable is Eq. (4.51), $u_i^* s_i = 0$, $i = 1$ to m. Multiplying the equation by s_i, we get $u_i^* s_i^2 = 0$. Now substituting $s_i^2 = -g_i(\mathbf{x}^*)$, we get $u_i^* g_i(\mathbf{x}^*) = 0$, $i = 1$ to m. This way the slack variable is eliminated from the equation and the switching condition of Eq. (4.51) can be written as $u_i^* g_i(\mathbf{x}^*) = 0$, $i = 1$ to m. These conditions can be used to define various cases as $u_i^* = 0$ or $g_i = 0$ (instead of $s_i = 0$). Table 5.1 gives the KKT conditions of Theorem 4.6 in the alternate form without the slack variables, and Examples 5.1 and 5.2 provide illustrations of their use.

EXAMPLE 5.1 USE OF THE ALTERNATE FORM OF THE KKT CONDITIONS

Minimize

$$f(x, y) = (x - 10)^2 + (y - 8)^2 \tag{a}$$

subject to

$$g_1 = x + y - 12 \le 0 \tag{b}$$

$$g_2 = x - 8 \le 0 \tag{c}$$

Solution

Since the problem is already expressed in the standard form, there is no need to perform any transformations to convert the problem to the standard form. The KKT conditions are

1. Lagrangian function definition of Eq. (5.1):

$$L = (x-10)^2 + (y-8)^2 + u_1(x+y-12) + u_2(x-8) \tag{d}$$

2. Gradient condition of Eq. (5.2):

$$\begin{aligned} \frac{\partial L}{\partial x} &= 2(x-10) + u_1 + u_2 = 0 \\ \frac{\partial L}{\partial y} &= 2(y-8) + u_1 = 0 \end{aligned} \tag{e}$$

3. Feasibility check of Eq. (5.3):

$$g_1 \le 0,\ g_2 \le 0 \tag{f}$$

4. Switching conditions of Eq. (5.4):

$$u_1 g_1 = 0,\ u_2 g_2 = 0 \tag{g}$$

5. Nonnegativity of Lagrange multipliers of Eq. (5.5):

$$u_1, u_2 \ge 0 \tag{h}$$

6. Regularity check.

The switching conditions of Eq. (g) give the following four cases:

1. $u_1 = 0$, $u_2 = 0$ (both g_1 and g_2 inactive)
2. $u_1 = 0$, $g_2 = 0$ (g_1 inactive, g_2 active)
3. $g_1 = 0$, $u_2 = 0$ (g_1 active, g_2 inactive)
4. $g_1 = 0$, $g_2 = 0$ (both g_1 and g_2 active)

Case 1: $u_1 = 0$, $u_2 = 0$ (both g_1 and g_2 inactive)

Eq. (e) give the solution as, $x = 10$, $y = 8$. Checking feasibility of this point gives $g_1 = 6 > 0$, $g_2 = 2 > 0$; thus both constraints are violated and so this case does not give any feasible candidate minimum point.

Case 2: $u_1 = 0$, $g_2 = 0$ (g_1 inactive, g_2 active)

$g_2 = 0$ gives $x = 8$. Eq. (e) give $y = 8$ and $u_2 = 4$. At the point (8, 8), $g_1 = 4 > 0$, which is a violation. Thus the point (8, 8) is infeasible and this case also does not give any feasible candidate minimum points.

Case 3: $g_1 = 0$, $u_2 = 0$ (g_1 active, g_2 inactive)

Eq. (e) and $g_1 = 0$ give $x = 7$, $y = 5$, $u_1 = 6 > 0$. Checking feasibility, $g_2 = -1 < 0$, which is satisfied. Since there is only one active constraint, the question of linear dependence of gradients of active constraints does not arise; therefore, regularity condition is satisfied. Thus point (7, 5) satisfies all the KKT necessary conditions.

Case 4: $g_1 = 0$, $g_2 = 0$ (both g_1 and g_2 active)

The case $g_1 = 0$, $g_2 = 0$ gives $x = 8$, $y = 4$. Eq. (e) give $u_1 = 8$, $u_2 = -4 < 0$, which is a violation of the necessary conditions. Therefore, this case also does not give any candidate minimum points.

It may be checked that this is a convex programming problem since constraints are linear and the cost function is convex. Therefore, the point obtained in Case 3 is indeed a global minimum point according to the convexity results of Section 4.8.

EXAMPLE 5.2 CHECK FOR KKT NECESSARY CONDITIONS

An optimization problem has one equality constraint h and one inequality constraint g. Check the KKT necessary conditions at what is believed to be the minimum point using the following information:

$$h = 0,\ \ g = 0,\quad \nabla f = (2, 3, 2),\quad \nabla h = (1, -1, 1),\quad \nabla g = (-1, -2, -1) \tag{a}$$

Solution

At the candidate minimum point, the gradients of h and g are linearly independent, so the given point is regular. To check linear independence, we form a linear combination of ∇h and ∇g and set it to zero (refer to Appendix A for this check):

$$c_1 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ -2 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{b}$$

where c_1 and c_2 are the parameters of linear combination. If $c_1 = 0$ and $c_2 = 0$ is the only solution for the linear system in Eq. (b), then the vectors are linearly independent. In the linear system in Eq. (b), the first and the third equations are the same; the determinant of the coefficient matrix of the first two equations is -3; therefore, the only solution is $c_1 = 0$ and $c_2 = 0$.

The KKT conditions for the problem are

$$\begin{gathered} \nabla L = \nabla f + v\nabla h + u\nabla g = 0 \\ h = 0,\quad g \le 0,\quad ug = 0,\quad u \ge 0 \end{gathered} \tag{c}$$

Substituting for ∇f, ∇h, and ∇g in $\nabla L = 0$, we get the following three equations:

$$2 + v - u = 0,\quad 3 - v - 2u = 0,\quad 2 + v - u = 0 \tag{d}$$

These are three equations in two unknowns; however, only two of them are linearly independent. Solving for u and v, we get $u = 5/3 \ge 0$ and $v = -1/3$. Thus, all of the KKT necessary conditions are satisfied.

5.2 IRREGULAR POINTS

In all of the examples that have been considered thus far, it is implicitly assumed that conditions of the KKT Theorem 4.6 or the Lagrange Theorem 4.5 are satisfied. In particular, we have assumed that **x*** is a *regular point* of the feasible design space. That is, gradients of all the active constraints at **x*** are linearly independent (ie, they are neither parallel to each other, nor can any gradient be expressed as a linear combination of others). It must be realized that necessary conditions are *applicable only if the assumption of the regularity* of **x*** is satisfied. To show that the necessary conditions are not applicable if **x*** is not a regular point, we consider Example 5.3.

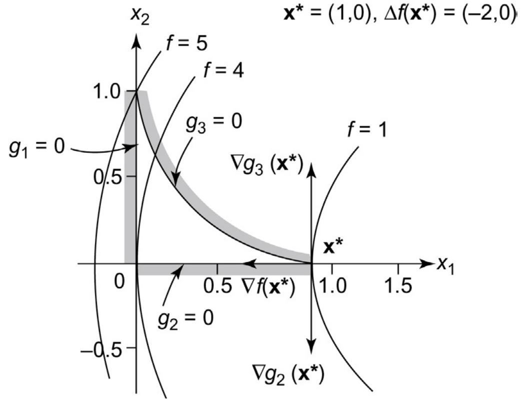

FIGURE 5.1 **Graphical solution for Example 5.3: irregular optimum point.**

EXAMPLE 5.3 CHECK FOR KKT CONDITIONS AT IRREGULAR POINTS

Minimize

$$f(x_1, x_2) = x_1^2 + x_2^2 - 4x_1 + 4 \tag{a}$$

subject to

$$g_1 = -x_1 \le 0 \tag{b}$$

$$g_2 = -x_2 \le 0 \tag{c}$$

$$g_3 = x_2 - (1 - x_1)^3 \le 0 \tag{d}$$

Check if the minimum point (1, 0) satisfies the KKT necessary conditions (McCormick, 1967).

Solution

The graphical solution, shown in Fig. 5.1, gives the global minimum for the problem at **x*** = (1, 0). Let us see if the solution satisfies the KKT necessary conditions:

1. Lagrangian function definition of Eq. (5.1):

$$L = x_1^2 + x_2^2 - 4x_1 + 4 + u_1(-x_1) + u_2(-x_2) + u_3[x_2 - (1 - x_1)^3] \tag{e}$$

2. Gradient condition of Eq. (5.2):

$$\begin{aligned} \frac{\partial L}{\partial x_1} &= 2x_1 - 4 - u_1 + u_3(3)(1 - x_1)^2 = 0 \\ \frac{\partial L}{\partial x_2} &= 2x_2 - u_2 + u_3 = 0 \end{aligned} \tag{f}$$

3. Feasibility check of Eq. (5.3):

$$g_i \le 0, \ \ i = 1,2,3 \tag{g}$$

4. Switching conditions of Eq. (5.4):

$$u_i g_i = 0, \ \ i = 1,2,3 \tag{h}$$

5. Nonnegativity of Lagrange multipliers of Eq. (5.5):

$$u_i \ge 0, \ \ i = 1,2,3 \tag{i}$$

6. Regularity check.

At $\mathbf{x}^* = (1, 0)$ the first constraint (g_1) is inactive and the second and third constraints are active. The switching conditions in Eq. (h) identify the case as $u_1 = 0, g_2 = 0, g_3 = 0$. Substituting the solution into Eqs. (f), we find that the first equation gives $-2 = 0$ and therefore, it is not satisfied. Thus, the KKT necessary conditions are not satisfied at the minimum point.

This apparent *contradiction* can be resolved by checking the regularity condition at the minimum point $\mathbf{x}^* = (1, 0)$. The gradients of the active constraints g_2 and g_3 are given as

$$\nabla g_2 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}; \quad \nabla g_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{j}$$

These vectors are not linearly independent. They are along the same line but in opposite directions, as seen in Fig. 5.1. Thus $\mathbf{x}^*$ is not a regular point of the feasible set. Since this is assumed in the KKT conditions, their use is invalid here. Note also that the geometrical interpretation of the KKT conditions of Eq. (4.53) is violated; that is, for the present example, ∇f at (1, 0) cannot be written as a linear combination of the gradients of the active constraints g_2 and g_3. Actually, ∇f is normal to both ∇g_2 and ∇g_3, as seen in the Fig. 5.1; therefore it cannot be expressed as their linear combination.

Note that for some problems irregular points can be obtained as a solution to the KKT conditions; however, in such cases, the Lagrange multipliers of the active constraints cannot be guaranteed to be unique. Also, the constraint variation sensitivity result of Section 4.7 may or may not be applicable to some values of the Lagrange multipliers.

5.3 SECOND-ORDER CONDITIONS FOR CONSTRAINED OPTIMIZATION

Solutions to the first-order necessary conditions are candidate local minimum designs. In this section, we will discuss second-order necessary and sufficiency conditions for constrained optimization problems. As in the unconstrained case, *second-order information* about the functions at the candidate point $\mathbf{x}^*$ will be used to determine if the point is indeed a local minimum. Recall for the unconstrained problem that the local sufficiency of Theorem 4.4 requires the quadratic part of Taylor's expansion for the function at $\mathbf{x}^*$ to be positive for all nonzero design changes $\mathbf{d}$. *In the constrained case, we must also consider active constraints at* $\mathbf{x}^*$ *to determine feasible changes* $\mathbf{d}$. We will consider only the points $\mathbf{x} = \mathbf{x}^* + \mathbf{d}$ in the neighborhood of $\mathbf{x}^*$ that satisfy the active constraint equations.

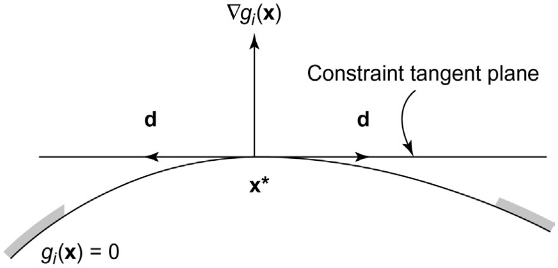

FIGURE 5.2 **Directions d used in second-order conditions.**

Any $\mathbf{d} \neq 0$ *satisfying active constraints to the first order must be in the constraint tangent hyperplane* (Fig. 5.2). Such $\mathbf{d}$'s are then orthogonal to the gradients of the active constraints since constraint gradients are normal to the constraint tangent hyperplane. Therefore, the dot product of $\mathbf{d}$ with each of the active constraint gradients ∇h_i and ∇g_i must be zero; that is, $\nabla h_i^T \mathbf{d} = 0$ and $\nabla g_i^T \mathbf{d} = 0$. These equations are used to determine directions $\mathbf{d}$ that define a feasible region around the point $\mathbf{x}^*$. Note that only active inequality constraints ($g_i = 0$) are used in determining $\mathbf{d}$. The situation is depicted in Fig. 5.2 for one inequality constraint.

To derive the second-order conditions, we write Taylor's expansion of the Lagrange function and consider only those $\mathbf{d}$ that satisfy the preceding conditions. $\mathbf{x}^*$ is then a local minimum point if the second-order term of Taylor's expansion is positive for all $\mathbf{d}$ in the constraint tangent hyperplane. This is then the sufficient condition for an isolated local minimum point. As a necessary condition the second-order term must be nonnegative. We summarize these results in Theorems 5.1 and 5.2.

THEOREM 5.1

Second-Order Necessary Conditions for General Constrained Problems

Let $\mathbf{x}^*$ satisfy the first-order KKT necessary conditions for the general optimum design problem. Define the Hessian of the Lagrange function L at $\mathbf{x}^*$ as

$$\nabla^2 L = \nabla^2 f + \sum\nolimits_{i=1}^{p} v_j^* \nabla^2 h_i + \sum\nolimits_{j=1}^{m} u_j^* \nabla^2 g_j \tag{5.6}$$

Let there be nonzero feasible directions, $\mathbf{d} \neq \mathbf{0}$, satisfying the following linear systems at the point $\mathbf{x}^*$:

$$\nabla h_i^T \mathbf{d} = 0; \; i = 1 \text{ to } p \tag{5.7}$$

$$\nabla g_i^T \mathbf{d} = 0 \text{ for all active inequalities (ie, for those } j \text{ with } g_j(\mathbf{x}^*) = 0) \tag{5.8}$$

Then, if $\mathbf{x}^*$ is a local minimum point for the optimum design problem, it must be true that

$$Q \geq 0 \;\; \text{where} \;\; Q = \mathbf{d}^{\mathrm{T}} \nabla^2 L(\mathbf{x}^*) \mathbf{d} \tag{5.9}$$

Note that any point that does not satisfy the second-order necessary conditions cannot be a local minimum point.

THEOREM 5.2

Sufficient Conditions for General Constrained Problems

Let **x*** satisfy the first-order KKT necessary conditions for the general optimum design problem. Define the Hessian of the Lagrange function L at **x*** as shown in Eq. (5.6). Define nonzero feasible directions, $\mathbf{d} \neq 0$, as solutions to the linear systems:

$$\nabla h_i^T \mathbf{d} = 0; \ \ i = 1 \text{ to } p \tag{5.10}$$

$$\nabla g_j^T \mathbf{d} = 0 \text{ for all those active inequalities with } u_j^* > 0 \tag{5.11}$$

Also let $\nabla g_j^T \mathbf{d} \leq 0$ for those active inequalities with $u_j^* = 0$. If

$$Q > 0, \quad \text{where } Q = \mathbf{d}^{\mathrm{T}} \nabla^2 L(\mathbf{x}^*) \mathbf{d} \tag{5.12}$$

then **x*** is an *isolated local minimum* point (*isolated* means that there are no other local minimum points in the neighborhood of **x***).

Insights for Second-Order Conditions

1. Note first the difference in the conditions for the directions **d** in Eq. (5.8) for the necessary condition and Eq. (5.11) for the sufficient condition. In Eq. (5.8), all active inequalities with nonnegative multipliers are included, whereas in Eq. (5.11) only those active inequalities with a positive multiplier are included.
2. Eqs. (5.10) and (5.11) simply say that the dot product of vectors ∇h_i and **d** and ∇g_j (having $u_j^* > 0$) and **d** should be zero. Thus, only the **d** orthogonal to the gradients of equality and active inequality constraints with $u_j^* > 0$ are considered. Stated differently, only **d** in the tangent hyperplane to the active constraints at the candidate minimum point are considered.
3. Eq. (5.12) says that the Hessian of the Lagrangian must be positive definite for all **d** lying in the constraint tangent hyperplane. Note that ∇h_i, ∇g_j, and $\nabla^2 L$ are calculated at the candidate local minimum points **x*** satisfying the KKT necessary conditions.
4. It should also be emphasized that if the inequality in Eq. (5.12) is not satisfied (ie, $Q \not> 0$), we cannot conclude that **x*** is not a local minimum. It may still be a local minimum but not an isolated one. Note also that the theorem cannot be used for any **x*** if its assumptions are not satisfied. In that case, we cannot draw any conclusions for the point **x***.
5. It is important to note that if matrix $\nabla^2 L(\mathbf{x}^*)$ is negative definite or negative semidefinite then the second-order necessary condition in Eq. (5.9) for a local minimum is violated and **x*** cannot be a local minimum point.
6. It is also important to note that if $\nabla^2 L(\mathbf{x}^*)$ is positive definite (ie, Q in Eq. (5.12) is positive for any $\mathbf{d} \neq \mathbf{0}$) then **x*** satisfies the sufficiency condition for an isolated local minimum

and no further checks are needed. The reason is that if $\nabla^2 L(\mathbf{x}^*)$ is positive definite, then it is positive definite for those $\mathbf{d}$ that satisfy Eqs. (5.10) and (5.11). However, *if* $\nabla^2 L(\mathbf{x}^*)$ *is not positive definite (ie, it is positive semidefinite or indefinite), then we cannot conclude that* $\mathbf{x}^*$ *is not an isolated local minimum.* We must calculate $\mathbf{d}$ to satisfy Eqs. (5.10) and (5.11) and carry out the sufficiency test given in Theorem 5.2. This result is summarized in Theorem 5.3.

THEOREM 5.3

Strong Sufficient Condition

Let $\mathbf{x}^*$ satisfy the first-order KKT necessary conditions for the general optimum design problem. Define Hessian $\nabla^2 L(\mathbf{x}^*)$ for the Lagrange function at $\mathbf{x}^*$ as shown in Eq. (5.6). Then, if $\nabla^2 L(\mathbf{x}^*)$ is positive definite, $\mathbf{x}^*$ is an isolated minimum point.

7. One case arising in some applications needs special mention. This occurs when the total number of active constraints (with at least one inequality) at the candidate minimum point $\mathbf{x}^*$ is equal to the number of independent design variables; that is, there are no design degrees of freedom at the candidate minimum point. Since $\mathbf{x}^*$ satisfies the KKT necessary conditions, the gradients of all the active constraints are linearly independent. Thus, the only solution for the system of Eqs. (5.10) and (5.11) is $\mathbf{d} = \mathbf{0}$ and Theorem 5.2 cannot be used. However, since $\mathbf{d} = \mathbf{0}$ is the only solution, there are no feasible directions in the neighborhood that can reduce the cost function any further. Thus, the point $\mathbf{x}^*$ is indeed a local minimum for the cost function (see also the definition of a local minimum in Section 4.1.1). We consider Examples 5.4–5.6 to illustrate the use of second-order conditions of optimality.

EXAMPLE 5.4 CHECK FOR SECOND-ORDER CONDITIONS 1

Check the second-order condition for Example 4.30:

Minimize

$$f(\mathrm{x}) = \frac{1}{3}x^3 - \frac{1}{2}(b+c)x^2 + bcx + f_0 \tag{a}$$

subject to

$$a \le x \le d \tag{b}$$

where $0 < a < b < \mathrm{c} < d$ and f_0 are specified constants.

Solution

There is only one constrained candidate local minimum point, $x = a$. Since there is only one design variable and one active constraint, the condition $\nabla g_1 \bar{d} = 0$ of Eq. (5.11) gives $\bar{d} = 0$ as the only solution (note that $\bar{d}$ is used as a direction for sufficiency check since d is used as a constant in the example). Therefore, Theorem 5.2 cannot be used for a sufficiency check. Also note that at $x = a$, $d^2L/dx^2 = (a - b) + (a - c)$, which is always negative, so we cannot use curvature of the Lagrangian function to check the sufficiency condition (strong sufficient Theorem 5.3). However, from Fig. 4.16 we observe that $x = a$ is indeed an isolated local minimum point.

From this example, we see that if the number of active inequality constraints is equal to the number of independent design variables and all other KKT conditions are satisfied, then the candidate point is indeed a local minimum point.

EXAMPLE 5.5 CHECK FOR SECOND-ORDER CONDITIONS 2

Consider the optimization problem of Example 4.31:

Minimize

$$f(\mathrm{x}) = x_1^2 + x_2^2 - 3x_1x_2 \tag{a}$$

subject to

$$g(\mathrm{x}) = x_1^2 + x_2^2 - 6 \le 0 \tag{b}$$

Check for sufficient conditions for the candidate minimum points.

Solution

From solution of Example 4.31, the points satisfying KKT necessary conditions are

$$\text{(i) } \mathbf{x}^* = (0,0),\ u^* = 0;\quad \text{(ii) } \mathbf{x}^* = (\sqrt{3},\sqrt{3}),\ u^* = \frac{1}{2};\quad \text{(iii) } \mathbf{x}^* = (-\sqrt{3},-\sqrt{3}),\ u^* = \frac{1}{2} \tag{c}$$

It was observed in Example 4.31 and Fig. 4.17 that the point (0, 0) did not satisfy the sufficiency condition and that the other two points did satisfy it. Those geometrical observations will be mathematically verified using the second-order optimality conditions.

The Hessian matrices for the cost and constraint functions are

$$\nabla^2 f = \begin{bmatrix} 2 & -3 \\ -3 & 2 \end{bmatrix},\quad \nabla^2 g = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \tag{d}$$

By the method of Appendix A, eigenvalues of $\nabla^2 g$ are $\lambda_1 = 2$ and $\lambda_2 = 2$. Since both eigenvalues are positive, the function g is convex, and so the feasible set defined by $g(\mathbf{x}) \le 0$ is convex by Theorem 4.9. However, since eigenvalues of $\nabla^2 f$ are -1 and 5, f is not convex. Therefore, it cannot be classified as a convex programming problem and sufficiency cannot be shown by the convexity Theorem 4.11. We must resort to the sufficiency Theorem 5.2.

The Hessian of the Lagrangian function is given as

$$\nabla^2 L = \nabla^2 f + u\nabla^2 g = \begin{bmatrix} 2+2u & -3 \\ -3 & 2+2u \end{bmatrix} \tag{e}$$

1. For the first point $\mathbf{x}^* = (0, 0)$, $u^* = 0$, $\nabla^2 L$ becomes $\nabla^2 f$ (the constraint $g(\mathbf{x}) \le 0$ is inactive). In this case, the problem is unconstrained and the local sufficiency requires $\mathbf{d}^T\nabla^2 f(\mathbf{x}^*)\mathbf{d} > 0$ for all $\mathbf{d}$.

Or $\nabla^2 f$ should be positive definite at $\mathbf{x}^*$. Since both eigenvalues of $\nabla^2 f$ are not positive, we conclude that the aforementioned condition is not satisfied. Therefore, $\mathbf{x}^* = (0, 0)$ does not satisfy the second-order sufficiency condition for a local minimum. Note that since $\lambda_1 = -1$ and $\lambda_2 = 5$, the matrix $\nabla^2 f$ is indefinite at $\mathbf{x}^*$. The point $\mathbf{x}^* = (0, 0)$, then, *violates the second-order necessary condition* of Theorem 4.4 requiring $\nabla^2 f$ to be at least positive semidefinite at the candidate local minimum point. Thus, $\mathbf{x}^* = (0, 0)$ cannot be a local minimum point. This agrees with graphical observation made in Example 4.31.

2. At points $\mathbf{x}^* = (\sqrt{3}, \sqrt{3}),\ u^* = \dfrac{1}{2}$ and $\mathbf{x}^* = (-\sqrt{3}, -\sqrt{3}),\ u^* = \dfrac{1}{2}$,

$$\nabla^2 L = \nabla^2 f + u\nabla^2 g = \begin{bmatrix} 2+2u & -3 \\ -3 & 2+2u \end{bmatrix} = \begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix} \tag{f}$$

$$\nabla g = \pm(2\sqrt{3}, 2\sqrt{3}) = \pm 2\sqrt{3}(1,1) \tag{g}$$

It may be checked that $\nabla^2 L$ is not positive definite at either of the two points. Therefore, we cannot use Theorem 5.3 to conclude that $\mathbf{x}^*$ is an isolated local minimum point. We must find $\mathbf{d}$ satisfying Eqs. (5.10) and (5.11). If we let $\mathbf{d} = (d_1, d_2)$, then $\nabla g^T \mathbf{d} = 0$ gives

$$\pm 2\sqrt{3}\begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = 0; \quad \text{or} \quad d_1 + d_2 = 0 \tag{h}$$

Thus, $d_1 = -d_2 = c$, where $c \neq 0$ is an arbitrary constant, and a $\mathbf{d} \neq 0$ satisfying $\nabla g^T \mathbf{d} = 0$ is given as $\mathbf{d} = c(1, -1)$. The sufficiency condition of Eq. (5.12) gives

$$Q = \mathbf{d}^T(\nabla^2 L)\mathbf{d} = c[1 -1]\begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix} c\begin{bmatrix} 1 \\ -1 \end{bmatrix} = 12c^2 > 0 \ \text{ for } \ c \neq 0 \tag{i}$$

The points $\mathbf{x}^* = (\sqrt{3}, \sqrt{3})$ and $\mathbf{x}^* = (-\sqrt{3}, -\sqrt{3})$ satisfy the sufficiency condition of Eq. (5.12). Therefore, they are isolated local minimum points, as was observed graphically in Example 4.31 and Fig. 4.17. We see for this example that $\nabla^2 L$ is not positive definite at $\mathbf{x}^*$, but $\mathbf{x}^*$ is still an isolated minimum point.

Note that since f is continuous and the feasible set is closed and bounded, we are guaranteed the existence of a global minimum by the Weierstrass Theorem 4.1. Also we have examined every possible point satisfying necessary conditions. Therefore, we must conclude by elimination that $\mathbf{x}^* = (\sqrt{3}, \sqrt{3})$ and $\mathbf{x}^* = (-\sqrt{3}, -\sqrt{3})$ are global minimum points. The value of the cost function for both points is $f(\mathbf{x}^*) = -3$.

EXAMPLE 5.6 CHECK FOR SECOND-ORDER CONDITIONS 3

Consider Example 4.32:

Minimize

$$f(x_1, x_2) = x_1^2 + x_2^2 - 2x_1 - 2x_2 + 2 \tag{a}$$

subject to

$$g_1 = -2x_1 - x_2 + 4 \le 0 \tag{b}$$

$$g_2 = -x_1 - 2x_2 + 4 \le 0 \tag{c}$$

Check the second-order conditions for the candidate minimum point.

Solution

From Example 4.32, the KKT necessary conditions are satisfied for the point

$$x_1^* = \frac{4}{3}, \quad x_2^* = \frac{4}{3}, \quad u_1^* = \frac{2}{9}, \quad u_2^* = \frac{2}{9} \tag{d}$$

Since all the constraint functions are linear, the feasible set S is convex. The Hessian of the cost function is positive definite. Therefore, it is also convex and the problem is convex and by Theorem 4.11,

$$x_1^* = \frac{4}{3}, \quad x_2^* = \frac{4}{3}$$

satisfies sufficiency conditions for a global minimum with the cost function as $f(\mathbf{x}^*) = \dfrac{2}{9}$

Note that local sufficiency cannot be shown by the method of Theorem 5.2. The reason is that the conditions of Eq. (5.11) give two equations in two unknowns:

$$-2d_1 - d_2 = 0, \; -d_1 - 2d_2 = 0 \tag{e}$$

This is a homogeneous system of equations with a nonsingular coefficient matrix. Therefore, its only solution is $d_1 = d_2 = 0$. Thus, we cannot find a $\mathbf{d} \neq \mathbf{0}$ for use in the condition of Eq. (5.12), and Theorem 5.2 cannot be used. However, we have seen in the foregoing and in Fig. 4.18 that the point is actually an isolated global minimum point. Since it is a two-variable problem and two inequality constraints are active at the KKT point, the condition for a local minimum is satisfied.

5.4 SECOND-ORDER CONDITIONS FOR THE RECTANGULAR BEAM DESIGN PROBLEM

The rectangular beam design problem is formulated and graphically solved in Fig. 3.11 in Section 3.8. The KKT necessary conditions are written and solved in Section 4.9.2. Several points that satisfy the KKT conditions are obtained. It is seen from the graphical representation of the problem in Fig. 3.11 that all of these points are global minima for the problem; however, none of the points is an isolated local minimum. Let us show that the second-order sufficiency condition of Theorem 5.2 will not be satisfied for any of these points.

Cases 3, 5, and 6 in Section 4.9.2 gave solutions that satisfy all the KKT necessary conditions. Cases 5 and 6 had two active constraints with g_1 having Lagrange multiplier value of zero; however, only the constraint with positive multiplier needs to be considered in Eq. (5.11). The sufficiency Theorem 5.2 requires only constraints with $u_i > 0$ to be considered in calculating the feasible directions for use in Eq. (5.12). Therefore, only the g_2 constraint needs to be included in the check for sufficiency conditions. Thus, we see that *all the three cases have the same sufficiency check*.

We need to calculate Hessians of the cost function and the second constraint:

$$\nabla^2 f = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \nabla^2 g_2 = \frac{(2.25\times10^5)}{b^3d^3}\begin{bmatrix} 2d^2 & bd \\ bd & 2b^2 \end{bmatrix} \tag{a}$$

Since $bd = (1.125 \times 10^5)$, $\nabla^2 g_2$ becomes

$$\nabla^2 g_2 = 2\begin{bmatrix} \dfrac{2}{b^2} & (1.125\times10^5)^{-1} \\ (1.125\times10^5)^{-1} & \dfrac{2}{d^2} \end{bmatrix} \tag{b}$$

The Hessian of the Lagrangian is given as

$$\nabla^2 L = \nabla^2 f + u_2\nabla^2 g_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + 2(56,250)\begin{bmatrix} \dfrac{2}{b^2} & (1.125\times10^5)^{-1} \\ (1.125\times10^5)^{-1} & \dfrac{2}{d^2} \end{bmatrix} \tag{c}$$

$$\nabla^2 L = \begin{bmatrix} \dfrac{(2.25\times10^5)}{b^2} & 2 \\ 2 & \dfrac{(2.25\times10^5)}{d^2} \end{bmatrix} \tag{d}$$

Since the determinant of $\nabla^2 L$ is 0 for $bd = (1.125 \times 10^5)$, the matrix is only positive semidefinite. Therefore, the Strong Sufficiency Theorem 5.3 cannot be used to show the sufficiency of $\mathbf{x}^*$. We must check the sufficiency condition of Eq. (5.12). In order to do that, we must find directions $\mathbf{y}$ (since d is used as a design variable, we use $\mathbf{y}$ instead of $\mathbf{d}$) satisfying Eq. (5.11). The gradient of g_2 is given as

$$\nabla g_2 = \left[\frac{-(2.25\times10^5)}{b^2d}, \quad \frac{-(2.25\times10^5)}{bd^2}\right] \tag{e}$$

The feasible directions $\mathbf{y}$ at the point $bd = (1.125 \times 10^5)$ are given by $\nabla g_2{}^T\mathbf{y} = 0$, as

$$\frac{1}{b}y_1 + \frac{1}{d}y_2 = 0, \quad \text{or} \quad y_2 = -\frac{d}{b}y_1 \tag{f}$$

Therefore, vector **y** is given as $\mathbf{y} = (1, -d/b)c$, where $c = y_1$ is any constant. Using $\nabla^2 L$ and **y**, Q of Eq. (5.12) is given as

$$Q = \mathbf{y}^T \nabla^2 L \mathbf{y} = 0 \tag{g}$$

Thus, the sufficiency condition of Theorem 5.2 is not satisfied. The points satisfying $bd = (1.125 \times 10^5)$ are not isolated local minimum points. This is, of course, true from Fig. 3.11. Note, however, that since $Q = 0$, the second-order necessary condition of Theorem 5.1 is satisfied for Case 3. Theorem 5.1 cannot be used to check the second-order necessary conditions for solutions to Cases 5 and 6 since there are two active constraints for this two-variable problem; therefore, there are no nonzero **y** vectors.

It is important to note that this problem does not satisfy the conditions for a convex programming problem and all of the points satisfying KKT conditions do not satisfy the sufficiency condition for an isolated local minimum. Yet all of the points are actually global minimum points. Two conclusions can be drawn from this example:

1. **Global minimum points can be obtained for problems that cannot be classified as convex programming problems. We cannot show global optimality of a point unless we find all of the local minimum points in the closed and bounded feasible set (the Weierstrass Theorem 4.1).**
2. **If second-order sufficiency condition is not satisfied, the only conclusion we can draw is that the candidate point is not an isolated local minimum. It may have many local optima in the neighborhood, and they may all be actually global minimum points.**

5.5 DUALITY IN NONLINEAR PROGRAMMING

Given a nonlinear programming problem, there is another nonlinear programming problem closely associated with it. The former is called the *primal problem*, and the latter is called the *dual problem*. Under certain convexity assumptions, the primal and dual problems have the same optimum objective function values and therefore, it is possible to solve the primal problem indirectly by solving the dual problem. As a by-product of one of the duality theorems, we obtain the *saddle point necessary conditions*.

Duality has played an important role in the development of optimization theory and numerical methods. Development of the duality theory requires *assumptions* about the convexity of the problem. However, to be broadly applicable, the theory should require a minimum of convexity assumptions. This leads to the concept of local convexity and to the *local duality theory*.

In this section, we will present only the local duality. The theory can be used to develop computational methods for solving optimization problems. We will see in chapter: More on Numerical Methods for Unconstrained Optimum Design, that it can be used to develop the so-called *augmented Lagrangian methods*.

5.5.1 Local Duality: Equality Constraints Case

For sake of developing the *local duality theory*, we consider the equality-constrained problem first.

Problem E

Find an n-vector $\mathbf{x}$ to

Minimize

$$f(\mathbf{x}) \tag{5.13}$$

subject to

$$h_i(\mathbf{x}) = 0; \quad i = 1 \text{ to } p \tag{5.14}$$

Later on we will extend the theory to both equality- and inequality-constrained problems. The theory we are going to present is sometimes called the *strong duality* or *Lagrangian duality*. We assume that functions f and h_i are twice continuously differentiable. We will first define a dual function associated with Problem E and study its properties. Then we will define the dual problem associated with Problem E.

To present the duality results for Problem E the following notation is used.

The Lagrangian function:

$$L(\mathbf{x}, \mathbf{v}) = f(\mathbf{x}) + \sum_{i=1}^{p} v_i h_i = f(\mathbf{x}) + (\mathbf{v} \bullet \mathbf{h}) \tag{5.15}$$

The Hessian of the Lagrangian function with respect to $\mathbf{x}$:

$$\mathbf{H}_x(\mathbf{x}, \mathbf{v}) = \frac{\partial^2 L}{\partial \mathbf{x}^2} = \frac{\partial^2 f(\mathbf{x})}{\partial \mathbf{x}^2} + \sum_{i=1}^{p} v_i \frac{\partial^2 h_i}{\partial \mathbf{x}^2} \tag{5.16}$$

The gradient matrix of equality constraints:

$$\mathbf{N} = \left[\frac{\partial h_j}{\partial x_i}\right]_{n \times p} \tag{5.17}$$

In these equations, $\mathbf{v}$ is the p-dimensional Lagrange multiplier vector for the equality constraints.

Let $\mathbf{x}^*$ be a local minimum of Problem E that is also a regular point of the feasible set. Then there exists a unique Lagrange multiplier v_i^* for each constraint such that the first-order necessary condition is met:

$$\frac{\partial L(\mathbf{x}^*, \mathbf{v}^*)}{\partial \mathbf{x}} = 0, \quad \text{or} \quad \frac{\partial f(\mathbf{x}^*)}{\partial \mathbf{x}} + \sum_{i=1}^{p} v_i^* \frac{\partial h_i(\mathbf{x}^*)}{\partial \mathbf{x}} = 0 \tag{5.18}$$

For development of the local duality theory, we make the assumption that the Hessian of the Lagrangian function $\mathbf{H}_x(\mathbf{x}^*, \mathbf{v}^*)$ at the minimum point $\mathbf{x}^*$ is positive definite. This assumption guarantees that the *Lagrangian* of Eq. (5.15) is *locally convex* at $\mathbf{x}^*$. This also satisfies the sufficiency condition for $\mathbf{x}^*$ to be an isolated local minimum of Problem E. With this assumption, the point $\mathbf{x}^*$ is not only a local minimum of Problem E, it is also a local minimum for the unconstrained problem:

$$\underset{\mathbf{x}}{\text{minimize}}\ L(\mathbf{x}, \mathbf{v}^*) \quad \text{or} \quad \underset{\mathbf{x}}{\text{minimize}} \left(f(\mathbf{x}) + \sum_{i=1}^{p} v_i^* h_i \right) \tag{5.19}$$

where $\mathbf{v}^*$ is a vector of Lagrange multipliers at $\mathbf{x}^*$. The necessary and sufficient conditions for the aforementioned unconstrained problem are the same as for the constrained Problem E (with $\mathbf{H}_x(\mathbf{x}^*, \mathbf{v}^*)$ being positive definite). In addition for any $\mathbf{v}$ *sufficiently close* to $\mathbf{v}^*$, the Lagrange function $L(\mathbf{x}, \mathbf{v})$ will have a local minimum at a point $\mathbf{x}$ near $\mathbf{x}^*$. Now we will establish the condition that $\mathbf{x}(\mathbf{v})$ exists and is a differentiable function of $\mathbf{v}$.

The necessary condition at the point $(\mathbf{x}, \mathbf{v})$ in the vicinity of $(\mathbf{x}^*, \mathbf{v}^*)$ is given as

$$\frac{\partial L(\mathbf{x}, \mathbf{v})}{\partial \mathbf{x}} = \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}} + \sum_{i=1}^{p} v_i \frac{\partial h_i}{\partial \mathbf{x}} = \mathbf{0}, \quad \text{or} \quad \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}} + \mathbf{N}\mathbf{v} = \mathbf{0} \tag{5.20}$$

Since $\mathbf{H}_x(\mathbf{x}^*, \mathbf{v}^*)$ is positive definite, it is nonsingular. Also because of this positive definiteness, $\mathbf{H}_x(\mathbf{x}, \mathbf{v})$ is positive definite in the vicinity of $(\mathbf{x}^*, \mathbf{v}^*)$ and thus nonsingular. This is a generalization of a theorem from calculus: *If a function is positive at a point, it is positive in a neighborhood of that point*. Note that $\mathbf{H}_x(\mathbf{x}, \mathbf{v})$ is also the Jacobian of the necessary conditions of Eq. (5.20) with respect to $\mathbf{x}$. Therefore, Eq. (5.20) has a solution $\mathbf{x}$ near $\mathbf{x}^*$ when $\mathbf{v}$ is near $\mathbf{v}^*$. Thus, locally there is a unique correspondence between $\mathbf{v}$ and $\mathbf{x}$ through a solution to the unconstrained problem:

$$\underset{\mathbf{x}}{\text{minimize}}\ L(\mathbf{x}, \mathbf{v}) \quad \text{or} \quad \underset{\mathbf{x}}{\text{minimize}} \left[f(\mathbf{x}) + \sum_{i=1}^{p} v_i h_i \right] \tag{5.21}$$

Furthermore, for a given $\mathbf{v}$, $\mathbf{x}(\mathbf{v})$ is a differentiable function of $\mathbf{v}$ (by the implicit functions theorem of calculus).

Dual Function

Near $\mathbf{v}^*$, we define the dual function $\phi(\mathbf{v})$ by the equation

$$\phi(\mathbf{v}) = \underset{\mathbf{x}}{\text{minimize}}\ L(\mathbf{x}, \mathbf{v}) \quad \text{or} \quad \underset{\mathbf{x}}{\text{minimize}} \left[f(\mathbf{x}) + \sum_{i=1}^{p} v_i h_i \right] \tag{5.22}$$

where the minimum is taken locally with respect to $\mathbf{x}$ near $\mathbf{x}^*$.

Dual Problem

$$\underset{\mathbf{v}}{\text{maximize}}\ \phi(\mathbf{v}) \tag{5.23}$$

With this definition of the dual function we can show that locally the original constrained Problem E is equivalent to unconstrained *local maximization of the dual function* $\phi(\mathbf{v})$ with respect to **v**. Thus, we can establish equivalence between a constrained problem in **x** and an unconstrained problem in **v**. To establish the duality relation, we must prove two lemmas.

LEMMA 5.1

The gradient of the dual function $\phi(\mathbf{v})$ is given as

$$\frac{\partial \phi(\mathbf{v})}{\partial \mathbf{v}} = \mathbf{h}(\mathbf{x}(\mathbf{v})) \tag{5.24}$$

Proof

Let $\mathbf{x}(\mathbf{v})$ represent a local minimum for the Lagrange function

$$L(\mathbf{x}, \mathbf{v}) = f(\mathbf{x}) + (\mathbf{v} \bullet \mathbf{h}) \tag{5.25}$$

Therefore, the dual function can be explicitly written from Eq. (5.22) as

$$\phi(\mathbf{v}) = [f(\mathbf{x}(\mathbf{v})) + (\mathbf{v} \bullet \mathbf{h}(\mathbf{x}(\mathbf{v})))] \tag{5.26}$$

where $\mathbf{x}(\mathbf{v})$ is a solution of the necessary condition in Eq. (5.20).

Now, differentiating $\phi(\mathbf{v})$ in Eq. (5.26) with respect to **v**, and using the fact that $\mathbf{x}(\mathbf{v})$ is a differentiable function of **v**, we get

$$\frac{\partial \phi(\mathbf{x}(\mathbf{v}))}{\partial \mathbf{v}} = \frac{\partial \phi(\mathbf{v})}{\partial \mathbf{v}} + \frac{\partial \mathbf{x}(\mathbf{v})}{\partial \mathbf{v}}\frac{\partial \phi}{\partial \mathbf{x}} = \mathbf{h}(\mathbf{x}(\mathbf{v})) + \frac{\partial \mathbf{x}(\mathbf{v})}{\partial \mathbf{v}}\frac{\partial L}{\partial \mathbf{x}} \tag{5.27}$$

where $\dfrac{\partial \mathbf{x}(\mathbf{v})}{\partial \mathbf{v}}$ is a $p \times n$ matrix. But $\partial L/\partial \mathbf{x}$ in Eq. (5.27) is zero because $\mathbf{x}(\mathbf{v})$ minimizes the Lagrange function of Eq. (5.25). This proves the result of Eq. (5.24).

Lemma 5.1 is of practical importance because it shows that the gradient of the dual function is quite simple to calculate. Once the dual function is evaluated by minimization with respect to **x**, the corresponding $\mathbf{h}(\mathbf{x})$, which is the gradient of $\phi(\mathbf{v})$, can be evaluated without any further calculation.

LEMMA 5.2

The Hessian of the dual function is given as

$$\mathbf{H}_v = \frac{\partial^2 \phi(\mathbf{v})}{\partial \mathbf{v}^2} = -\mathbf{N}^T [\mathbf{H}_x(\mathbf{x})]^{-1} \mathbf{N} \tag{5.28}$$

Proof

Differentiate Eq. (5.24) with respect to **v** to obtain

$$\mathbf{H}_v = \frac{\partial}{\partial \mathbf{v}}\left\{\frac{\partial \phi(\mathbf{x}(\mathbf{v}))}{\partial \mathbf{v}}\right\} = \frac{\partial \mathbf{h}(\mathbf{x}(\mathbf{v}))}{\partial \mathbf{v}} = \frac{\partial \mathbf{x}(\mathbf{v})}{\partial \mathbf{v}}\mathbf{N} \tag{5.29}$$

To calculate $\frac{\partial \mathbf{x}(\mathbf{v})}{\partial \mathbf{v}}$, we differentiate the necessary condition of Eq. (5.20) with respect to $\mathbf{v}$ to obtain

$$\mathbf{N}^T + \frac{\partial \mathbf{x}(\mathbf{v})}{\partial \mathbf{v}} \mathbf{H}_x(\mathbf{x}) = \mathbf{0} \tag{5.30}$$

Solving for $\frac{\partial \mathbf{x}(\mathbf{v})}{\partial \mathbf{v}}$ from Eq. (5.30), we get

$$\frac{\partial \mathbf{x}(\mathbf{v})}{\partial \mathbf{v}} = -\mathbf{N}^T[\mathbf{H}_x(\mathbf{x})]^{-1} \tag{5.31}$$

Substituting Eq. (5.31) into Eq. (5.29) and using the fact that $\mathbf{h}(\mathbf{x}(\mathbf{v}))$ does not depend explicitly on $\mathbf{v}$, we obtain the result of Eq. (5.28), which was to be proved.

Since $[\mathbf{H}_x(\mathbf{x})]^{-1}$ is positive definite, and since $\mathbf{N}$ is of full column rank near $\mathbf{x}$, we have $\mathbf{H}_v(\mathbf{v})$, a $p \times p$ matrix (Hessian of $\phi(\mathbf{v})$), to be *negative definite*. This observation and the Hessian of $\phi(\mathbf{v})$ play a role in the analysis of dual methods.

THEOREM 5.4

Local Duality Theorem

For Problem E, let

$\mathbf{x}^*$ be a local minimum.
$\mathbf{x}^*$ be a regular point.
$\mathbf{v}^*$ be the Lagrange multipliers at $\mathbf{x}^*$.
$\mathbf{H}_x(\mathbf{x}^*, \mathbf{v}^*)$ be positive definite.

Then for the dual problem
Maximize

$$\phi(\mathbf{v}) \tag{5.32}$$

has a local solution at $\mathbf{v}^*$ with $\mathbf{x}^* = \mathbf{x}(\mathbf{v}^*)$. The maximum value of the dual function is equal to the minimum value of $f(\mathbf{x})$; that is,

$$\phi(\mathbf{v}^*) = f(\mathbf{x}^*) \tag{5.33}$$

Proof

Solution of the necessary conditions in Eq. (5.20) gives $\mathbf{x} = \mathbf{x}(\mathbf{v})$ for use in the definition of the dual function $\phi(\mathbf{v})$. Therefore at $\mathbf{v}^*$, $\mathbf{x}^* = \mathbf{x}(\mathbf{v}^*)$. Now, at $\mathbf{v}^*$, we have Lemma 5.1:

$$\frac{\partial \phi(\mathbf{v}^*)}{\partial \mathbf{v}} = \mathbf{h}(\mathbf{x}) = \mathbf{0} \tag{a}$$

Also, by Lemma 5.2, the Hessian of $\phi(\mathbf{v})$ is negative definite. Thus, $\mathbf{v}^*$ satisfies the first-order necessary and second-order sufficiency conditions for an unconstrained maximum point of $\phi(\mathbf{v})$.

Substituting $\mathbf{v}^*$ in the definition of $\phi(\mathbf{v})$ in Eq. (5.26), we get

$$\begin{aligned}\phi(\mathbf{v}^*) &= [f(\mathbf{x}(\mathbf{v}^*)) + (\mathbf{v}^* \bullet \mathbf{h}(\mathbf{x}(\mathbf{v}^*)))] \\ &= [f(\mathbf{x}^*) + (\mathbf{v}^* \bullet \mathbf{h}(\mathbf{x}^*))] \\ &= f(\mathbf{x}^*)\end{aligned} \tag{b}$$

which was to be proved.

EXAMPLE 5.7 SOLUTION TO THE DUAL PROBLEM

Consider the following problem in two variables; derive the dual of the problem and solve it: Minimize

$$f = -x_1 x_2 \tag{a}$$

subject to

$$(x_1 - 3)^2 + x_2^2 = 5 \tag{b}$$

Solution

Let us first solve the primal problem using the optimality conditions. The Lagrangian for the problem is given as

$$L = -x_1 x_2 + v[(x_1 - 3)^2 + x_2^2 - 5] \tag{c}$$

The first-order necessary conditions are

$$-x_2 + (2x_1 - 6)v = 0 \tag{d}$$

$$-x_1 + 2x_2 v = 0 \tag{e}$$

Together with the equality constraint in Eq. (b), these equations have a solution:

$$x_1^* = 4,\ x_2^* = 2,\ v^* = 1,\ f^* = -8 \tag{f}$$

The Hessian of the Lagrangian function is given as

$$\mathbf{H}_x(\mathbf{x}^*, \mathbf{v}^*) = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \tag{g}$$

Since this is a positive definite matrix, we conclude that the solution obtained satisfies second-order sufficiency conditions, and therefore, is an isolated local minimum.

Since $\mathbf{H}_x(\mathbf{x}^*, \mathbf{v}^*)$ is positive definite, we can apply the local duality theory near the solution point. Define a dual function as

$$\phi(\mathbf{v}) = \underset{\mathbf{x}}{\text{minimize}}\ L(\mathbf{x}, \mathbf{v}) \tag{h}$$

Solving Eqs. (d) and (e), we get x_1 and x_2 in terms of v, provided that

$$4v^2 - 1 \neq 0 \tag{i}$$

and

$$x_1 = \frac{12v^2}{4v^2 - 1}, \; x_2 = \frac{6v}{4v^2 - 1} \tag{j}$$

Substituting Eqs. (j) into Eq. (c), the dual function of Eq. (h) is given as

$$\phi(v) = \frac{4v + 4v^3 - 80v^5}{(4v^2 - 1)^2} \tag{k}$$

which is valid for $v \neq \pm \frac{1}{2}$. This $\phi(v)$ has a local maximum at $v^* = 1$. Substituting $v = 1$ in Eqs. (j), we get the same solution as in Eqs. (f). Note that $\phi(v^*) = -8$, which is the same as f^* in Eq. (f).

5.5.2 Local Duality: The Inequality Constraints Case

Consider the equality/inequality-constrained problem.

Problem P

In addition to the equality constraints in Problem E, we impose inequality constraints:

$$g_i(\mathbf{x}) \leq 0; \; i = 1 \text{ to } m \tag{5.34}$$

The feasible set S for Problem P is defined as

$$S = \{\mathrm{x} \mid h_i(\mathbf{x}) = 0, \, i = 1 \text{ to } p; \; g_j(\mathbf{x}) \leq 0, \, j = 1 \text{ to } m\} \tag{5.35}$$

The Lagrangian function is defined as

$$\begin{aligned} L(\mathbf{x}, \mathbf{v}, \mathbf{u}) &= f(\mathbf{x}) + \sum_{i=1}^{p} v_i h_i + \sum_{j=1}^{m} u_j g_j \\ &= f(\mathbf{x}) + (\mathbf{v} \bullet \mathbf{h}) + (\mathbf{u} \bullet \mathbf{g}); \quad u_j \geq 0, \, j = 1 \text{ to } m \end{aligned} \tag{5.36}$$

The dual function for Problem P is defined as

$$\phi(\mathbf{v}, \mathbf{u}) = \underset{\mathbf{x}}{\text{minimize}} \; L(\mathbf{x}, \mathbf{v}, \mathbf{u}); \; u_j \geq 0, \; j = 1 \text{ to } m \tag{5.37}$$

The dual problem is defined as

$$\underset{\mathbf{v}, \mathbf{u}}{\text{maximize}} \; \phi(\mathbf{v}, \mathbf{u}); \quad u_j \geq 0, \; j = 1 \text{ to } m \tag{5.38}$$

THEOREM 5.5

Strong Duality Theorem

Let the following apply:

$\mathbf{x}^*$ be a local minimum of Problem P.
$\mathbf{x}^*$ be a regular point.
$\mathbf{H}_x(\mathbf{x}^*, \mathbf{v}^*, \mathbf{u}^*)$ be positive definite.
$\mathbf{v}^*$, $\mathbf{u}^*$ be the Lagrange multipliers at the optimum point $\mathbf{x}^*$.

Then $\mathbf{v}^*$ and $\mathbf{u}^*$ solve the dual problem that is defined in Eq. (5.38) with $f(\mathbf{x}^*) = \phi(\mathbf{v}^*, \mathbf{u}^*)$ and $\mathbf{x}^* = \mathbf{x}(\mathbf{v}^*, \mathbf{u}^*)$.

If the assumption of the positive definiteness of $\mathbf{H}_x(\mathbf{x}^*, \mathbf{v}^*)$ is not made, we get the weak duality theorem.

THEOREM 5.6

Weak Duality Theorem

Let $\mathbf{x}$ be a feasible solution for Problem P and let $\mathbf{v}$ and $\mathbf{u}$ be the feasible solution for the dual problem that is defined in Eq. (5.38); thus, $h_i(\mathbf{x}) = 0,\ i = 1 \text{ to } p$, and $g_j(\mathbf{x}) \le 0 \text{ and } u_j \ge 0,\ j = 1 \text{ to } m$. Then

$$\phi(\mathbf{v}, \mathbf{u}) \le f(\mathbf{x}) \tag{5.39}$$

Proof

By definition

$$\begin{aligned} \phi(\mathbf{v}, \mathbf{u}) &= \underset{\mathbf{x}}{\text{minimize}}\ L(\mathbf{x}, \mathbf{v}, \mathbf{u}) \\ &= \underset{\mathbf{x}}{\text{minimize}}\ [f(\mathbf{x}) + (\mathbf{v} \bullet \mathbf{h}) + (\mathbf{u} \bullet \mathbf{g})] \\ &\le [f(\mathbf{x}) + (\mathbf{v} \bullet \mathbf{h}) + (\mathbf{u} \bullet \mathbf{g})] \le f(\mathbf{x}) \end{aligned}$$

since $u_i \ge 0$, $g_i(\mathbf{x}) \le 0$, and $u_i g_i = 0$ for $i = 1$ to m; and $h_i(\mathbf{x}) = 0$, $i = 1$ to p.

From Theorem 5.5, we obtain the following results:

1. Minimum $[f(\mathbf{x})$ with $\mathbf{x} \in S] \ge$ maximum $[\phi(\mathbf{v}, \mathbf{u})$ with $u_i \ge 0$, $i = 1$ to $m]$.
2. If $f(\mathbf{x}^*) = \phi(\mathbf{v}^*, \mathbf{u}^*)$ with $u_i \ge 0$, $i = 1$ to m and $\mathbf{x}^* \in S$, then $\mathbf{x}^*$ and $(\mathbf{v}^*, \mathbf{u}^*)$ solve the primal and dual problems, respectively.

3. If Minimum $[f(\mathbf{x})$ with $\mathbf{x} \in S] = -\infty$, then the dual is *infeasible*, and vice versa (ie, if dual is *infeasible*, the primal is *unbounded*).
4. If Maximum $[\phi(\mathbf{v}, \mathbf{u})$ with $u_i \geq 0$, $i = 1$ to $m] = \infty$, then the primal problem has *no feasible solution*, and vice versa (ie, if primal is *infeasible*, the dual is *unbounded*).

LEMMA 5.3

Lower Bound for Primal Cost Function

For any $\mathbf{v}$ and $\mathbf{u}$ with $u_i \geq 0$, $i = 1$ to m

$$\phi(\mathbf{v}, \mathbf{u}) \leq f(\mathbf{x}^*) \tag{5.40}$$

Proof

$\phi(\mathbf{v}, \mathbf{u}) \leq$ maximum $\phi(\mathbf{v}, \mathbf{u})$; $u_i \geq 0$, $i = 1$ to m

$$= \underset{\mathbf{v},\,\mathbf{u}}{\text{maximize}} \left\{ \underset{\mathbf{x}}{\text{minimize}} \, [f(\mathbf{x}) + (\mathbf{v} \bullet \mathbf{h}) + (\mathbf{u} \bullet \mathbf{g})];\, u_i \geq 0,\, i = 1 \text{ to } m \right\}$$

$$= \underset{\mathbf{v},\,\mathbf{u}}{\text{maximize}} \left\{ f(\mathbf{x}(\mathbf{v}, \mathbf{u})) + (\mathbf{v} \bullet \mathbf{h}) + (\mathbf{u} \bullet \mathbf{g}) \right\};\, u_i \geq 0,\, i = 1 \text{ to } m$$

$$= f(\mathbf{x}(\mathbf{v}^*, \mathbf{u}^*)) + (\mathbf{v}^* \bullet \mathbf{h}) + (\mathbf{u}^* \bullet \mathbf{g}) = f(\mathbf{x}^*)$$

Lemma 5.3 is quite useful for practical applications. It tells us how to find a lower bound on the optimum primal cost function. The dual cost function for arbitrary v_i, $i = 1$ to p and $u_i \geq 0$, $i = 1$ to m provides a *lower bound* for the primal cost function. Also for any $\mathbf{x} \in S$, $f(\mathbf{x})$ provides an *upper bound* for the optimum cost function.

Saddle Points

Let $L(\mathbf{x}, \mathbf{v}, \mathbf{u})$ be the Lagrange function. L has a saddle point at $\mathbf{x}^*$, $\mathbf{v}^*$, $\mathbf{u}^*$ subject to $u_i \geq 0$, $i = 1$ to m if

$$L(\mathbf{x}^*, \mathbf{v}, \mathbf{u}) \leq L(\mathbf{x}^*, \mathbf{v}^*, \mathbf{u}^*) \leq L(\mathbf{x}, \mathbf{v}^*, \mathbf{u}^*) \tag{5.41}$$

holds for all $\mathbf{x}$ near $\mathbf{x}^*$ and $(\mathbf{v}, \mathbf{u})$ near $(\mathbf{v}^*, \mathbf{u}^*)$ with $u_i \geq 0$ for $i = 1$ to m.

THEOREM 5.7

Saddle Point Theorem

For Problem P let all functions be twice continuously differentiable and let $L(\mathbf{x}, \mathbf{v}, \mathbf{u})$ be defined as

$$L(\mathbf{x}, \mathbf{v}, \mathbf{u}) = f(\mathbf{x}) + (\mathbf{v} \bullet \mathbf{h}) + (\mathbf{u} \bullet \mathbf{g}); \quad u_j \geq 0, \quad j = 1 \text{ to } m \tag{5.42}$$

Let $L(\mathbf{x}^*, \mathbf{v}^*, \mathbf{u}^*)$ exist with $u_i^* \ge 0$, $i = 1$ to m. Also let $\mathbf{H}_x(\mathbf{x}^*, \mathbf{v}^*, \mathbf{u}^*)$ be *positive definite*. Then $\mathbf{x}^*$ satisfying a suitable constraint qualification is a local minimum of Problem P if and only if $(\mathbf{x}^*, \mathbf{v}^*, \mathbf{u}^*)$ is a saddle point of the Lagrangian; that is,

$$L(\mathbf{x}^*, \mathbf{v}, \mathbf{u}) \le L(\mathbf{x}^*, \mathbf{v}^*, \mathbf{u}^*) \le L(\mathbf{x}, \mathbf{v}^*, \mathbf{u}^*) \tag{5.43}$$

for all $\mathbf{x}$ near $\mathbf{x}^*$ and all $(\mathbf{v}, \mathbf{u})$ near $(\mathbf{v}^*, \mathbf{u}^*)$, with $u_i \ge 0$ for $i = 1$ to m.

See Bazarra et al. (2006) for proof of Theorem 5.7.

EXERCISES FOR CHAPTER 5

5.1 *Answer true or false.*

1. A convex programming problem always has a unique global minimum point.
2. For a convex programming problem, KKT necessary conditions are also sufficient.
3. The Hessian of the Lagrange function must be positive definite at constrained minimum points.
4. For a constrained problem, if the sufficiency condition of Theorem 5.2 is violated, the candidate point $\mathbf{x}^*$ may still be a minimum point.
5. If the Hessian of the Lagrange function at $\mathbf{x}^*$, $\nabla^2 L(\mathbf{x}^*)$ is positive definite, the optimum design problem is convex.
6. For a constrained problem, the sufficient condition at $\mathbf{x}^*$ is satisfied if there are no feasible directions in a neighborhood of $\mathbf{x}^*$ along which the cost function reduces.

5.2 Formulate the problem of Exercise 4.84. Show that the solution point for the problem is not a regular point. Write KKT conditions for the problem, and study the implication of the irregularity of the solution point.

5.3 Solve the following problem using the graphical method:
Minimize $f(x_1, x_2) = (x_1 - 10)^2 + (x_2 - 5)^2$
subject to $x_1 + x_2 \le 12$, $x_1 \le 8$, $x_1 - x_2 \le 4$
Show that the minimum point does not satisfy the regularity condition. Study the implications of this situation.

Solve the following problems graphically. Check necessary and sufficient conditions for candidate local minimum points and verify them on the graph for the problem.

5.4 Minimize $f(x_1, x_2) = 4x_1^2 + 3x_2^2 - 5x_1x_2 - 8x_1$
subject to $x_1 + x_2 = 4$

5.5 Maximize $F(x_1, x_2) = 4x_1^2 + 3x_2^2 - 5x_1x_2 - 8x_1$
subject to $x_1 + x_2 = 4$

5.6 Minimize $f(x_1, x_2) = (x_1 - 2)^2 + (x_2 + 1)^2$
subject to $2x_1 + 3x_2 - 4 = 0$

5.7 Minimize $f(x_1, x_2) = 4x_1^2 + 9x_2^2 + 6x_2 - 4x_1 + 13$
subject to $x_1 - 3x_2 + 3 = 0$

5.8 Minimize $f(\mathbf{x}) = (x_1 - 1)^2 + (x_2 + 2)^2 + (x_3 - 2)^2$
subject to $2x_1 + 3x_2 - 1 = 0$
$x_1 + x_2 + 2x_3 - 4 = 0$

5.9 Minimize $f(x_1, x_2) = 9x_1^2 + 18x_1x_2 + 13x_2^2 - 4$
subject to $x_1^2 + x_2^2 + 2x_1 = 16$

5.10 Minimize $f(x_1, x_2) = (x_1 - 1)^2 + (x_2 - 1)^2$
subject to $x_1 + x_2 - 4 = 0$

5.11 Minimize $f(x_1, x_2) = 4x_1^2 + 3x_2^2 - 5x_1x_2 - 8$
subject to $x_1 + x_2 = 4$

5.12 Maximize $F(x_1, x_2) = 4x_1^2 + 3x_2^2 - 5x_1x_2 - 8$
subject to $x_1 + x_2 = 4$

5.13 Maximize $F(x_1, x_2) = 4x_1^2 + 3x_2^2 - 5x_1x_2 - 8$
subject to $x_1 + x_2 \le 4$

5.14 Minimize $f(x_1, x_2) = 4x_1^2 + 3x_2^2 - 5x_1x_2 - 8$
subject to $x_1 + x_2 \le 4$

5.15 Maximize $F(x_1, x_2) = 4x_1^2 + 3x_2^2 - 5x_1x_2 - 8x_1$
subject to $x_1 + x_2 \le 4$

5.16 Minimize $f(x_1, x_2) = (x_1 - 1)^2 + (x_2 - 1)^2$
subject to $x_1 + x_2 \ge 4$
$x_1 - x_2 - 2 = 0$

5.17 Minimize $f(x_1, x_2) = (x_1 - 1)^2 + (x_2 - 1)^2$
subject to $x_1 + x_2 = 4$
$x_1 - x_2 - 2 \ge 0$

5.18 Minimize $f(x_1, x_2) = (x_1 - 1)^2 + (x_2 - 1)^2$
subject to $x_1 + x_2 \ge 4$
$x_1 - x_2 \ge 2$

5.19 Minimize $f(x, y) = (x - 4)^2 + (y - 6)^2$
subject to $12 \ge x + y$
$x \ge 6, y \ge 0$

5.20 Minimize $f(x_1, x_2) = 2x_1 + 3x_2 - x_1^3 - 2x_2^2$
subject to $x_1 + 3x_2 \le 6$
$5x_1 + 2x_2 \le 10$
$x_1, x_2 \ge 0$

5.21 Minimize $f(x_1, x_2) = 4x_1^2 + 3x_2^2 - 5x_1x_2 - 8x_1$
subject to $x_1 + x_2 \le 4$

5.22 Minimize $f(x_1, x_2) = x_1^2 + x_2^2 - 4x_1 - 2x_2 + 6$
subject to $x_1 + x_2 \ge 4$

5.23 Minimize $f(x_1, x_2) = 2x_1^2 - 6x_1x_2 + 9x_2^2 - 18x_1 + 9x_2$
subject to $x_1 + 2x_2 \le 10$
$4x_1 - 3x_2 \le 20;\ x_i \ge 0;\ i = 1, 2$

5.24 Minimize $f(x_1, x_2) = (x_1 - 1)^2 + (x_2 - 1)^2$
subject to $x_1 + x_2 - 4 \le 0$

5.25 Minimize $f(x_1, x_2) = (x_1 - 1)^2 + (x_2 - 1)^2$
subject to $x_1 + x_2 - 4 \le 0$
$x_1 - x_2 - 2 \le 0$

5.26 Minimize $f(x_1, x_2) = (x_1 - 1)^2 + (x_2 - 1)^2$
subject to $x_1 + x_2 - 4 \le 0$
$2 - x_1 \le 0$

5.27 Minimize $f(x_1, x_2) = 9x_1^2 - 18x_1x_2 + 13x_2^2 - 4$
subject to $x_1^2 + x_2^2 + 2x_1 \ge 16$

5.28 Minimize $f(x_1, x_2) = (x_1 - 3)^2 + (x_2 - 3)^2$
subject to $x_1 + x_2 \le 4$
$x_1 - 3x_2 = 1$

5.29 Minimize $f(x_1, x_2) = x_1^3 - 16x_1 + 2x_2 - 3x_2^2$
subject to $x_1 + x_2 \le 3$

5.30 Minimize $f(x_1, x_2) = 3x_1^2 - 2x_1x_2 + 5x_2^2 + 8x_2$
subject to $x_1^2 - x_2^2 + 8x_2 \le 16$

5.31 Minimize $f(x, y) = (x - 4)^2 + (y - 6)^2$
subject to $x + y \le 12$
$x \le 6$
$x, y \ge 0$

5.32 Minimize $f(x, y) = (x - 8)^2 + (y - 8)^2$
subject to $x + y \le 12$
$x \le 6$
$x, y \ge 0$

5.33 Maximize $F(x, y) = (x - 4)^2 + (y - 6)^2$
subject to $x + y \le 12$
$6 \ge x$
$x, y \ge 0$

5.34 Maximize $F(r, t) = (r - 8)^2 + (t - 8)^2$
subject to $10 \ge r + t$
$t \le 5$
$r, t \ge 0$

5.35 Maximize $F(r, t) = (r - 3)^2 + (t - 2)^2$
subject to $10 \ge r + t$
$t \le 5$
$r, t \ge 0$

5.36 Maximize $F(r, t) = (r - 8)^2 + (t - 8)^2$
subject to $r + t \le 10$
$t \ge 0$
$r \ge 0$

5.37 Maximize $F(r, t) = (r - 3)^2 + (t - 2)^2$
subject to $10 \ge r + t$
$t \ge 5$
$r, t \ge 0$

5.38 Formulate and graphically solve Exercise 2.23 of the design of a cantilever beam using hollow circular cross-section. Check the necessary and sufficient conditions at the optimum point. The data for the problem are $P = 10$ kN; $l = 5$ m; modulus of elasticity, $E = 210$ GPa; allowable bending stress, $\sigma_a = 250$ MPa; allowable shear stress, $\tau_a = 90$ MPa; and mass density, $\rho = 7850$ kg/m^3; $0 \le R_o \le 20$ cm, and $0 \le R_i \le 20$ cm.

5.39 Formulate and graphically solve Exercise 2.24. Check the necessary and sufficient conditions for the solution points and verify them on the graph.

5.40 Formulate and graphically solve Exercise 3.28. Check the necessary and sufficient conditions for the solution points and verify them on the graph.

Find optimum solutions for the following problems graphically. Check necessary and sufficient conditions for the solution points and verify them on the graph for the problem.

5.41 A minimum weight tubular column design problem is formulated in Section 2.7 using mean radius R and thickness t as design variables. Solve the problem by imposing an additional constraint $R/t \le 50$ for the following data: $P = 50$ kN, $l = 5.0$ m, $E = 210$ GPa, $\sigma_a = 250$ MPa, and $\rho = 7850$ kg/m^3.

5.42 A minimum weight tubular column design problem is formulated in Section 2.7 using outer radius R_o and inner radius R_i as design variables. Solve the problem by imposing an additional constraint $0.5(R_o + R_i)/(R_o - R_i) \le 50$. Use the same data as in Exercise 5.41.

5.43 Solve the problem of designing a "can" formulated in Section 2.2.

5.44 Exercise 2.1

***5.45** Exercise 3.34

***5.46** Exercise 3.35

***5.47** Exercise 3.36

***5.48** Exercise 3.54

5.49 *Answer true or false.*

1. Candidate minimum points for a constrained problem that do not satisfy second-order sufficiency conditions can be global minimum designs.
2. Lagrange multipliers may be used to calculate the sensitivity coefficient for the cost function with respect to the right side parameters even if Theorem 4.7 cannot be used.
3. Relative magnitudes of the Lagrange multipliers provide useful information for practical design problems.

5.50 A circular tank that is closed at both ends is to be fabricated to have a volume of 250π m^3. The fabrication cost is found to be proportional to the surface area of the sheet metal needed for fabrication of the tank and is \$400/m^2. The tank is to be housed in a shed with a sloping roof which limits the height of the tank by the relation $H \le 8D$, where H is the height and D is the diameter of the tank. The problem is formulated as minimize $f(D, H) = 400(0.5\pi D^2 + \pi DH)$ subject to the constraints $\frac{\pi}{4}D^2H = 250\pi$, and $H \le 8D$. Ignore any other constraints.

1. Check for convexity of the problem.
2. Write KKT necessary conditions.
3. Solve KKT necessary conditions for local minimum points. Check sufficient conditions and verify the conditions graphically.
4. What will be the change in cost if the volume requirement is changed to 255π m^3 in place of 250π m^3?

5.51 A symmetric (area of member 1 is the same as area of member 3) three-bar truss problem is described in Section 2.10.

1. Formulate the minimum mass design problem treating A_1 and A_2 as design variables.
2. Check for convexity of the problem.
3. Write KKT necessary conditions for the problem.
4. Solve the optimum design problem using the data: $P = 50$ kN, $\theta = 30°$, $\rho = 7800$ kg/m^3, $\sigma_a = 150$ MPa. Verify the solution graphically and interpret the necessary conditions on the graph for the problem.
5. What will be the effect on the cost function if σ_a is increased to 152 MPa?

Formulate and solve the following problems graphically; check necessary and sufficient conditions at the solution points; verify the conditions on the graph for the problem and study the effect of variations in constraint limits on the cost function.

5.52 Exercise 2.1
5.53 Exercise 2.3
5.54 Exercise 2.4
5.55 Exercise 2.5
5.56 Exercise 2.9
5.57 Exercise 4.92
5.58 Exercise 2.12
5.59 Exercise 2.14
5.60 Exercise 2.23
5.61 Exercise 2.24
5.62 Exercise 5.41
5.63 Exercise 5.42
5.64 Exercise 5.43
5.65 Exercise 3.28
5.66 Exercise 3.34
***5.67** Exercise 3.35
***5.68** Exercise 3.36
***5.69** Exercise 3.39
***5.70** Exercise 3.40
***5.71** Exercise 3.41
***5.72** Exercise 3.46
***5.73** Exercise 3.47
***5.74** Exercise 3.48
***5.75** Exercise 3.49
***5.76** Exercise 3.50
***5.77** Exercise 3.51
***5.78** Exercise 3.52
***5.79** Exercise 3.53
***5.80** Exercise 3.54

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