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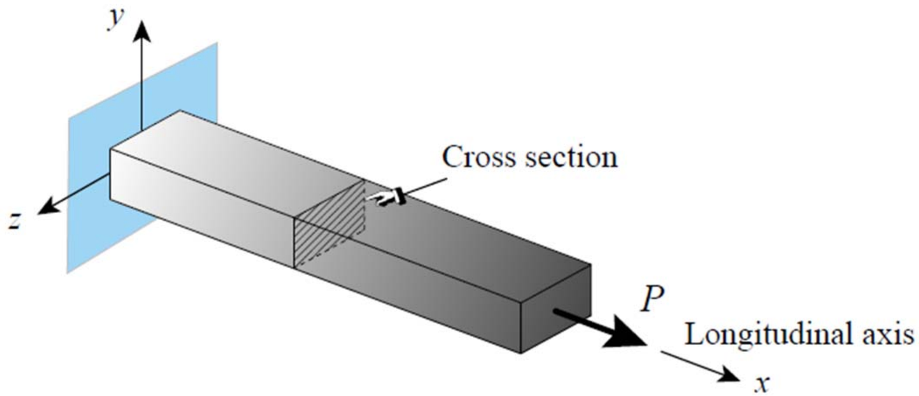
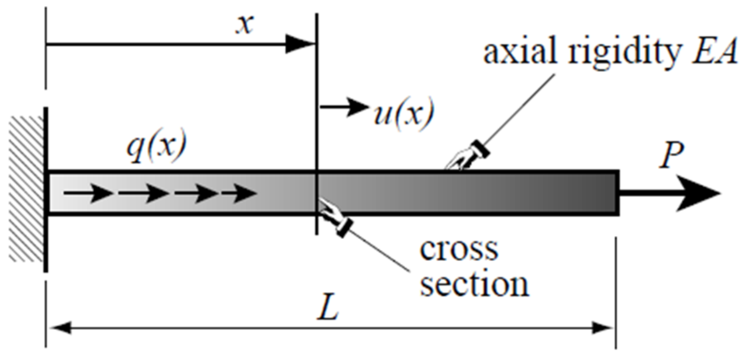
Bar Member

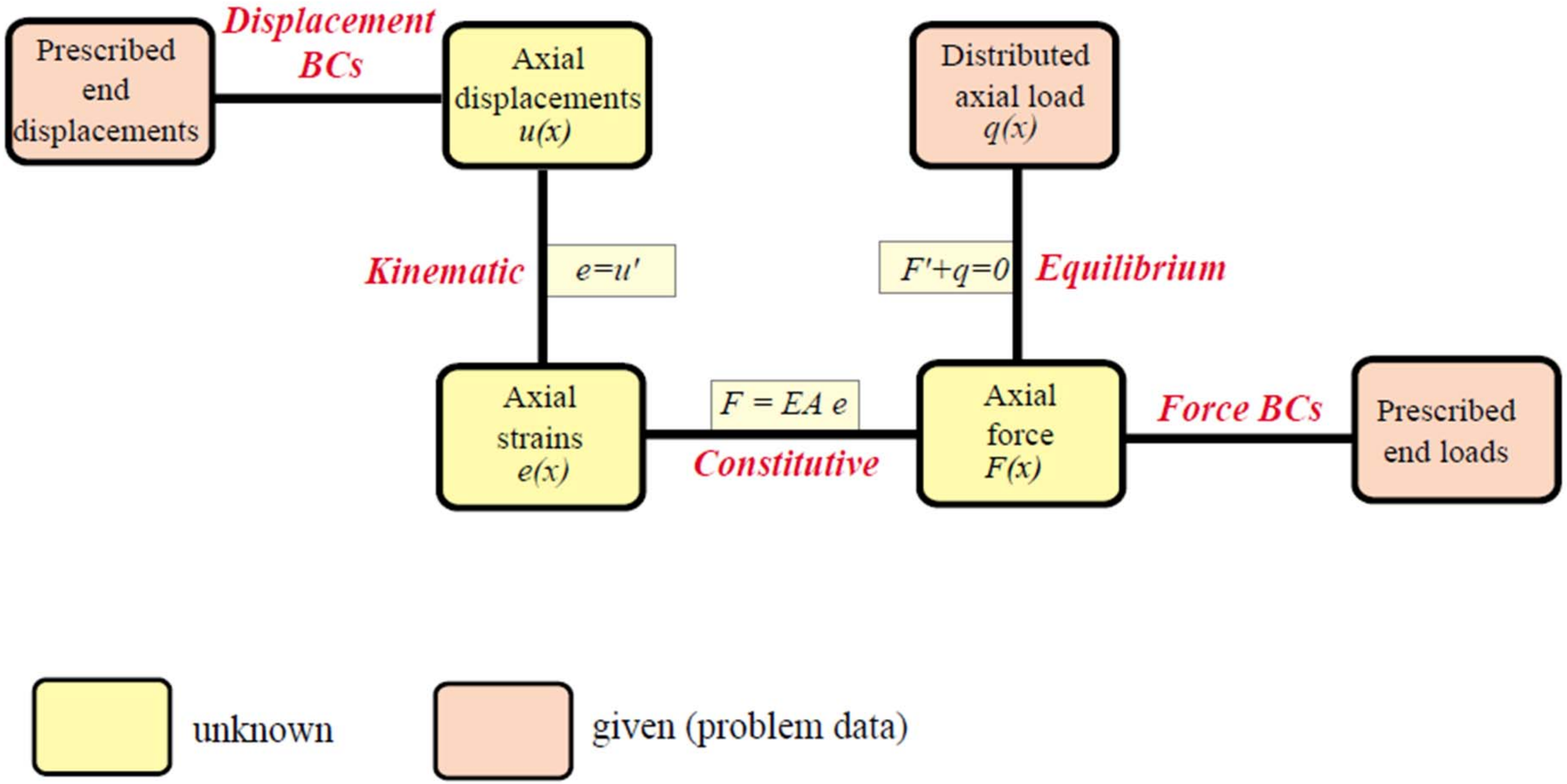
- Characteristics
  - One preferred (longitudinal dimension or axial) dimension
    - much larger than the other two (transverse) dimensions
    - cross section: intersection of a plane normal to the longitudinal dimension and the bar
  - Resist an internal axial force along its longitudinal dimension
- Modeling (truss)
  - cable, chain, rope
  - fictitious elements in penalty function method





Tonti Diagram of Governing Equations

- Straight bar: cross section may vary
- Linearly elastic material: Hooke's law
- Infinitesimal displacements and strains



Potential Energy of the Bar Member

Internal energy (=strain energy):

$$U = \frac{1}{2}\int_V \sigma e dV = \frac{1}{2}\int_0^L \sigma e\left(Adx\right)\left[=\frac{1}{2}\int_0^L Fedx\right] = \frac{1}{2}\int_0^L \left(EAu'\right)u'dx = \frac{1}{2}\int_0^L u'EAu'dx$$

External work: $W = \int_0^L qudx$

Total Potential Energy: $\Pi = U - W$

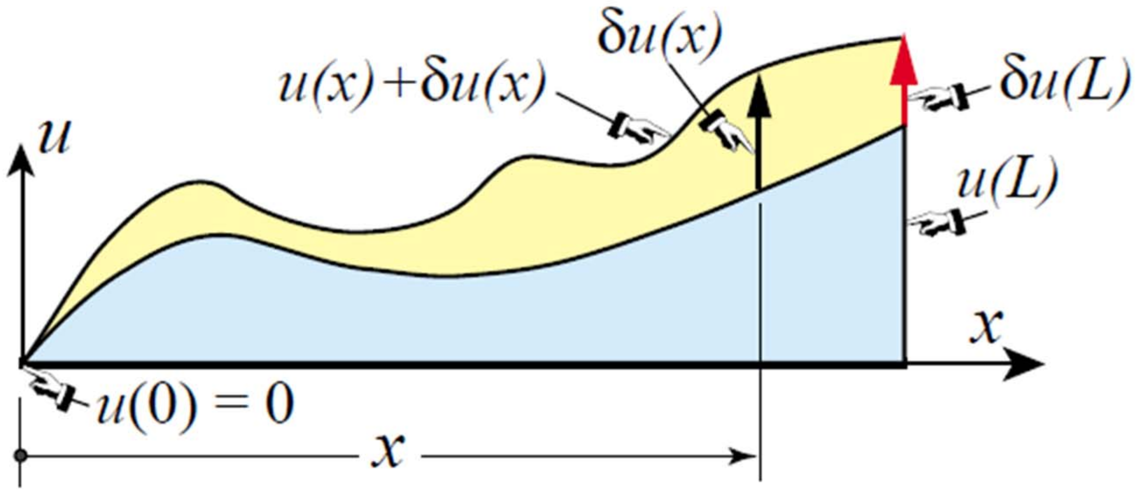
Minimum Total Potential Energy(MTPE) Principle:

actual displacement solution $u^*\left(x\right)$ that satisfies the governing equations is that which renders the TPE function $\Pi\left[u\right]$ stationary

$\delta\Pi = \delta U - \delta W = 0$ iff $u = u^*$

with respect to *admissible* variations $u = u^* + \delta u$ of the exact displacement solution $u^*\left(x\right)$

Concept of Kinematically Admissible Variation

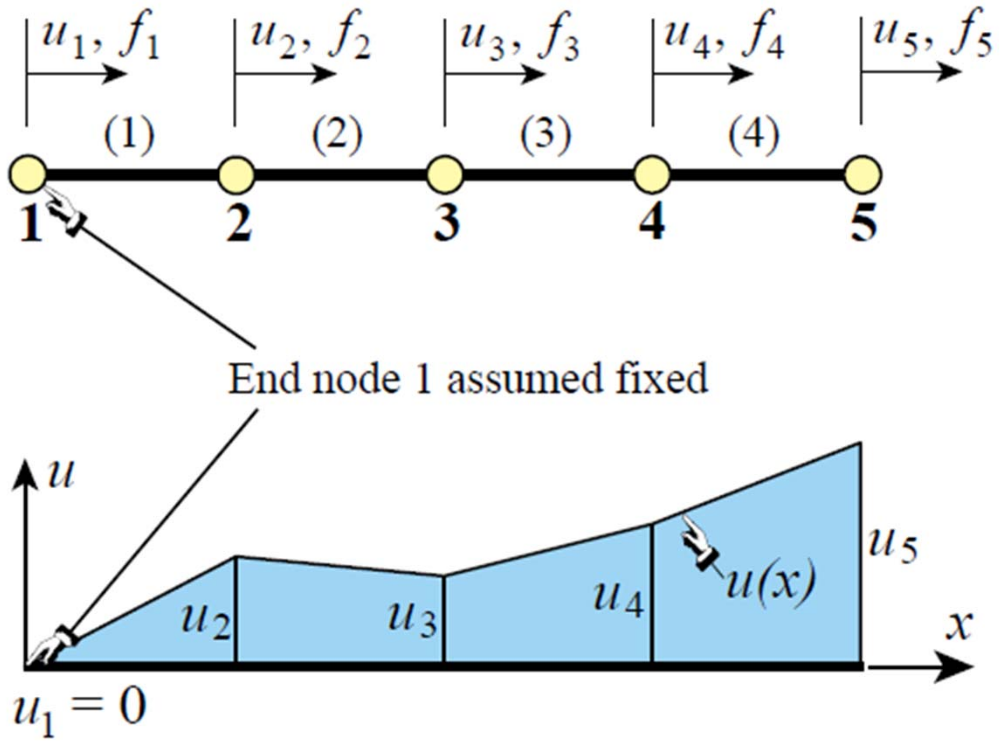


$\delta u(x)$ is kinematically admissible if $u(x)$ and $u(x)+\delta u(x)$

(i) are continuous over bar length, i.e. $u(x) \in C^0$ in $x \in [0, L]$

(ii) satisfy exactly displacement BC, in the figure, $u(0) = 0$

FEM Discretization and Displacement Trial Function



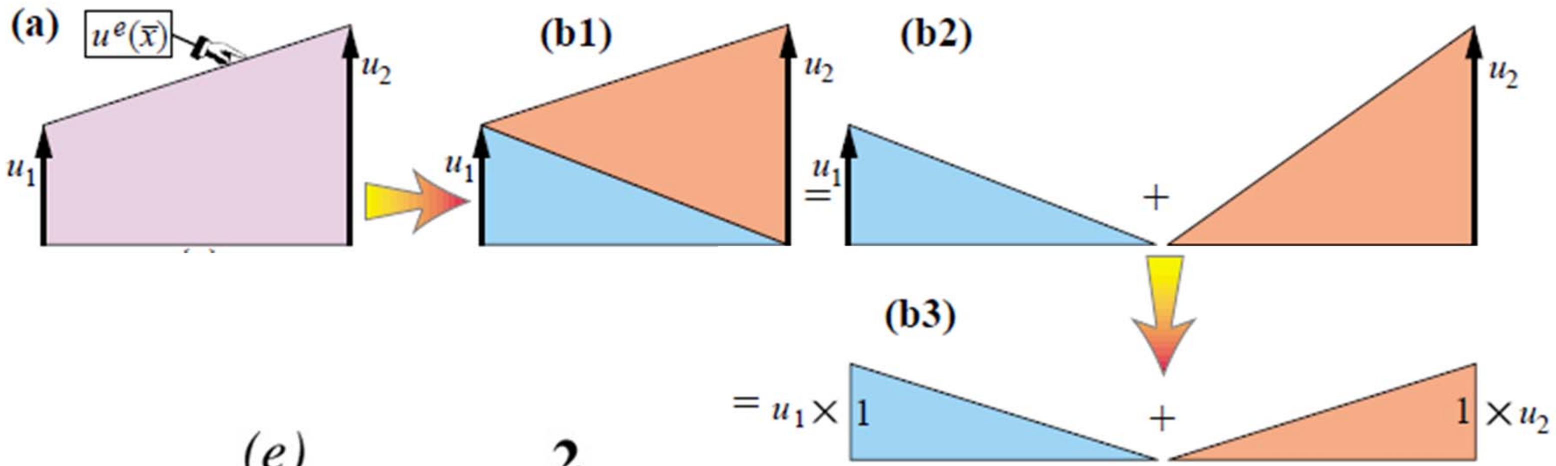
$$\delta\Pi = \delta U - \delta W = 0 \;\; \text{iff} \;\; u = u^* \left(\text{exact solution}\right)$$

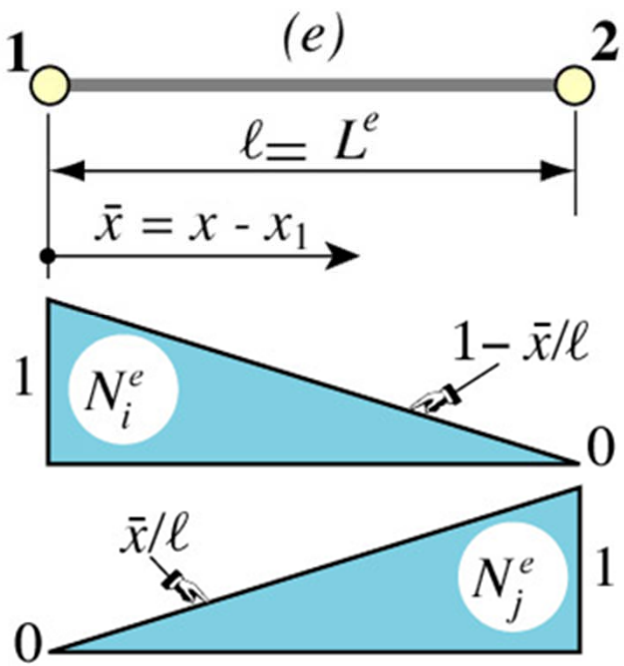
$$\Pi = \Pi^{(1)} + \Pi^{(2)} + \cdots + \Pi^{(N_e)}$$

$$\delta\Pi = \delta\Pi^{(1)} + \delta\Pi^{(2)} + \cdots + \delta\Pi^{(N_e)} = 0$$

$$\delta\Pi^e = \delta U^e - \delta W^e = 0$$

Element Shape Functions





$$u^*(x) \approx u^e(x) = N_1^e u_1^e + N_2^e u_2^e$$

$$= \begin{bmatrix} N_1^e & N_2^e \end{bmatrix} \begin{bmatrix} u_1^e \\ u_2^e \end{bmatrix} = \mathbf{N}\mathbf{u}^e$$

$$N_1^e = 1 - \frac{\bar{x}}{l} = 1 - \zeta, \; N_2^e = \frac{\bar{x}}{l} = \zeta$$

$$\zeta = \frac{x - x_1}{l} = \frac{\bar{x}}{l}$$

dimensionless (natural) coordinate

Finite Element Equation

$$\Pi^e = U^e - W^e \leftarrow \begin{cases} U^e = \dfrac{1}{2}\left(\mathbf{u}^e\right)^T \mathbf{K}^e \mathbf{u}^e \\ W^e = \left(\mathbf{u}^e\right)^T \mathbf{f}^e \end{cases}$$

$$\delta\Pi^e = \delta U^e - \delta W^e = \frac{1}{2}\left(\delta\mathbf{u}^e\right)^T \mathbf{K}^e \mathbf{u}^e + \frac{1}{2}\left(\mathbf{u}^e\right)^T \mathbf{K}^e \delta\mathbf{u}^e - \left(\delta\mathbf{u}^e\right)^T \mathbf{f}^e = 0$$

$$\xrightarrow{\mathbf{u}^e = \left(\mathbf{u}^e\right)^T,\ \delta\mathbf{u}^e = \left(\delta\mathbf{u}^e\right)^T} \left(\delta\mathbf{u}^e\right)^T \left[\mathbf{K}^e \mathbf{u}^e - \mathbf{f}^e\right] = 0$$

since $\delta\mathbf{u}^e$ is arbitrary, $\left[\ldots\right] = 0$

$\mathbf{K}^e \mathbf{u}^e = \mathbf{f}^e$ (element stiffness equations)

Bar Element Stiffness and Nodal Force Vector

$$[\text{strain-displacement}]\, e = \frac{du^e}{dx} = \left(u^e\right)' = \begin{bmatrix} \dfrac{dN_1^e}{dx} & \dfrac{dN_2^e}{dx} \end{bmatrix} \begin{bmatrix} u_1^e \\ u_2^e \end{bmatrix} = \frac{1}{l}\begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} u_1^e \\ u_2^e \end{bmatrix} = \mathbf{B}\mathbf{u}^e$$

$$[\text{internal energy}]\, U^e = \frac{1}{2}\int_0^l \left(u^e\right)' EA \left(u^e\right)' dx = \frac{1}{2}\int_0^1 \left(u^e\right)' EA \left(u^e\right)' l\, d\zeta$$

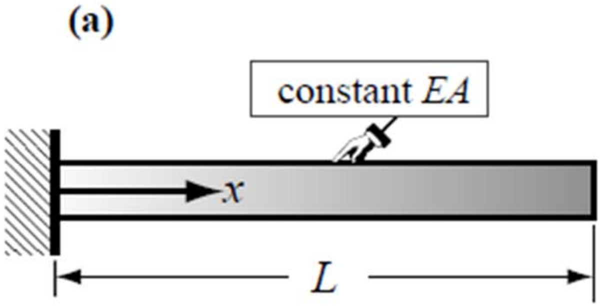
$$= \frac{1}{2}\int_0^1 \left(\mathbf{u}^e\right)^T \mathbf{B}^T EA\mathbf{B}\mathbf{u}^e l\, d\zeta = \frac{1}{2}\left(\mathbf{u}^e\right)^T \underbrace{\left[\int_0^1 EA\mathbf{B}^T\mathbf{B} l\, d\zeta\right]}_{\mathbf{K}^e} \left(\mathbf{u}^e\right) = \frac{1}{2}\left(\mathbf{u}^e\right)^T \mathbf{K}^e \left(\mathbf{u}^e\right)$$

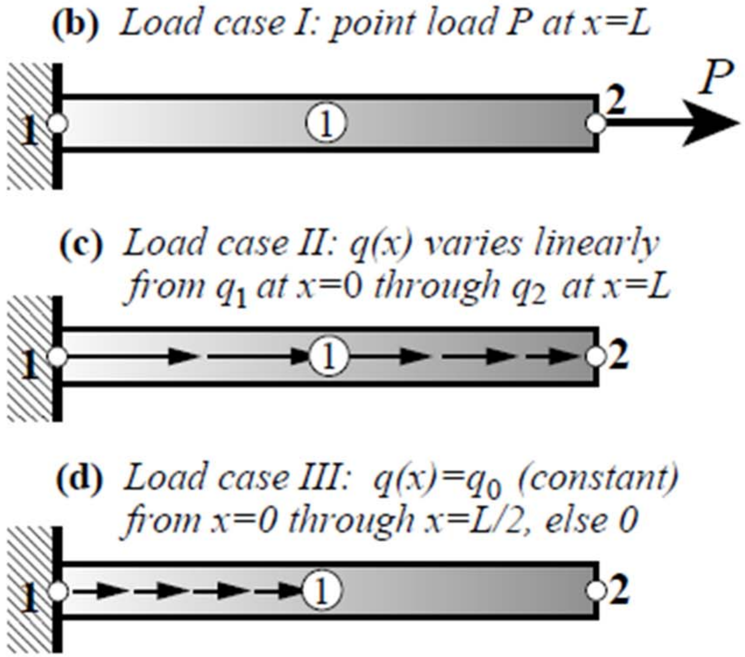
$$\mathbf{K}^e = \int_0^1 \frac{EA}{l^2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} l\, d\zeta \xrightarrow[\text{over the element}]{\text{if } EA \text{ is constant}} \mathbf{K}^e = \frac{EA}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$[\text{external work}]\, W^e = \int_0^l qu\, dx = \int_0^1 q\mathbf{N}^T\mathbf{u}^e l\, d\zeta = \left(\mathbf{u}^e\right)^T \underbrace{\int_0^1 q\begin{bmatrix} 1-\zeta \\ \zeta \end{bmatrix} l\, d\zeta}_{\mathbf{f}^e} = \left(\mathbf{u}^e\right)^T \mathbf{f}^e$$

$$\mathbf{f}^e = \int_0^1 q\begin{bmatrix} 1-\zeta \\ \zeta \end{bmatrix} l\, d\zeta \xrightarrow[\text{along the element}]{\text{if } q \text{ is constant}} \mathbf{f}^e = q\int_0^1 \begin{bmatrix} 1-\zeta \\ \zeta \end{bmatrix} l\, d\zeta = ql\begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} \leftarrow \text{EbE load lumping}$$

Example: Fixed-Free, Prismatic Bar (1)





$$q^{I}(x) = P\delta(L) \rightarrow f^{I} = \begin{bmatrix} 0 \\ P \end{bmatrix}$$

$$q^{II}(x) = q_1(1-\zeta) + q_2\varsigma \rightarrow f^{II} = \int_0^1 q^{II} \begin{bmatrix} 1-\zeta \\ \zeta \end{bmatrix} L d\varsigma = \frac{L}{6}\begin{bmatrix} 2q_1 + q_2 \\ q_1 + 2q_2 \end{bmatrix}$$

$$q^{III}(x) = q_0\left[H(x) - H\left(x - \frac{L}{2}\right)\right] \rightarrow f^{III} = \int_0^L q^{III} \begin{bmatrix} 1 - x/L \\ x/L \end{bmatrix} dx = \frac{q_0 L}{8}\begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

Example: Fixed-Free, Prismatic Bar (1)

$$q^{II}(x)=q_1(1-\zeta)+q_2\varsigma \rightarrow f^{II}=\int_0^1 q^{II}\begin{bmatrix}1-\zeta\\ \zeta\end{bmatrix}Ld\varsigma=\int_0^1\left[q_1(1-\zeta)+q_2\varsigma\right]\begin{bmatrix}1-\zeta\\ \zeta\end{bmatrix}Ld\varsigma=\int_0^1\begin{bmatrix}q_1\left(1-2\zeta+\zeta^2\right)+q_2\left(\varsigma-\zeta^2\right)\\ q_1\left(\varsigma-\zeta^2\right)+q_2\zeta^2\end{bmatrix}Ld\varsigma$$

$$=L\begin{Bmatrix}q_1\left[\zeta-\zeta^2+\dfrac{\zeta^3}{3}\right]_0^1+q_2\left[\dfrac{\zeta^2}{2}-\dfrac{\zeta^3}{3}\right]_0^1\\ q_1\left[\dfrac{\zeta^2}{2}-\dfrac{\zeta^3}{3}\right]_0^1+q_2\left[\dfrac{\zeta^3}{3}\right]_0^1\end{Bmatrix}=L\begin{bmatrix}q_1\dfrac{1}{3}+q_2\dfrac{1}{6}\\ q_1\dfrac{1}{6}+q_2\dfrac{1}{3}\end{bmatrix}=\frac{L}{6}\begin{bmatrix}2q_1+q_2\\ q_1+2q_2\end{bmatrix}$$

$$q^{III}(x)=q_0\left[H(x)-H\left(x-\frac{L}{2}\right)\right]\rightarrow f^{III}=\int_0^L q^{III}\begin{bmatrix}1-x/L\\ x/L\end{bmatrix}dx=\int_0^L q_0\left[H(x)-H\left(x-\frac{L}{2}\right)\right]\begin{bmatrix}1-x/L\\ x/L\end{bmatrix}dx$$

$$=q_0\begin{bmatrix}\int_0^L\left(1-\dfrac{x}{L}\right)dx-\int_{L/2}^L\left(1-\dfrac{x}{L}\right)dx\\ \int_0^L\dfrac{x}{L}dx-\int_{L/2}^L\dfrac{x}{L}dx\end{bmatrix}=q_0\begin{Bmatrix}\left[x-\dfrac{x^2}{2L}\right]_0^L-\left[x-\dfrac{x^2}{2L}\right]_{L/2}^L\\ \left[\dfrac{x^2}{2L}\right]_0^L-\left[\dfrac{x^2}{2L}\right]_{L/2}^L\end{Bmatrix}=q_0\begin{bmatrix}\dfrac{L}{2}-\dfrac{L}{8}\\ \dfrac{L}{8}\end{bmatrix}=\frac{q_0L}{8}\begin{bmatrix}3\\ 1\end{bmatrix}$$

Example: Fixed-Free, Prismatic Bar (2)

$$\frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \xrightarrow{u_1=0} \begin{cases} f^{I} = \begin{bmatrix} 0 \\ P \end{bmatrix} \rightarrow u_2 = \dfrac{PL}{EA} \\ f^{II} = \dfrac{L}{6}\begin{bmatrix} 2q_1 + q_2 \\ q_1 + 2q_2 \end{bmatrix} \rightarrow u_2 = \dfrac{(q_1 + 2q_2)L^2}{6EA} \\ f^{III} = \dfrac{q_0 L}{8}\begin{bmatrix} 3 \\ 1 \end{bmatrix} \rightarrow u_2 = \dfrac{q_0 L^2}{8EA} \end{cases}$$

[analytical solution]

$(EAu')' + q = 0$ with $u(0) = 0$ and

$$\begin{cases} F^{I}(L) = EAu'(L) = P,\ q = 0 \rightarrow u(x) = \dfrac{Px}{EA} \\ F^{II}(L) = EAu'(L) = 0,\ q = q_1\left(1 - \dfrac{x}{L}\right) + q_2\dfrac{x}{L} \rightarrow u(x) = \dfrac{x\left[3(q_1 + q_2)L - 3q_1 x + (q_1 - q_2)x^2/L\right]}{6EA} \\ F^{III}(L) = EAu'(L) = 0,\ q = q_0\left[\langle x\rangle^0 - \left\langle x - \dfrac{1}{2}L\right\rangle^0\right] \rightarrow u(x) = \dfrac{q_0}{2EA}\left(Lx - x^2 + \left\langle x - \dfrac{1}{2}L\right\rangle^2\right) \end{cases}$$

Three computed end deflections are *exact* ! Why?

Weak Forms

[Strong Form]

$$\begin{cases} \left(EAu'(x)\right)' + q(x) = 0 \xrightarrow{EA \text{ is constant}} EAu''(x) + q(x) = 0 \\ r(x) = \left(EAu'(x)\right)' + q(x) \xrightarrow{EA \text{ is constant}} r(x) = EAu''(x) + q(x) \\ r(x) = 0: \text{ at each point over the member span, } x \in [0, L] \end{cases}$$

[Weak Form]

$$\begin{cases} \text{relax the condition } \left(r(x) = 0 \text{ everywhere}\right) \rightarrow \text{satisfy in an average sense} \\ J = \int_0^L r(x) v(x)\, dx = 0 \\ v(x) = \begin{cases} \text{test function in a genenral mathematical context} \\ \text{weight(ing) function in the approximation method} \end{cases} \end{cases}$$

Example (1)

$$J=\int_0^L\left[EAu''(x)+q_0\right]v(x)\,dx=0\ \text{ with } u(0)=0,\ F(L)=EAu'(L)=0$$

[method 1]

$$\left.\begin{aligned} u(x)&=a_0+a_1x+a_2x^2 \rightarrow \text{trial function}\\ v(x)&=b_0+b_1x+b_2x^2 \rightarrow \text{weight function}\end{aligned}\right\}\xrightarrow[\text{bases}]{\text{same}}\text{Galerkin method}$$

(apply BCs *a posteri*)

$$J=\int_0^L\left[EA(2a_2)+q_0\right]\left(b_0+b_1x+b_2x^2\right)dx=\frac{L}{6}\left(6b_0+3b_1L+2b_2L^2\right)\left(2EAa_2+q_0\right)=0$$

$$\rightarrow u(x)=a_0+a_1x-\frac{q_0}{2EA}x^2\xrightarrow[F(L)=EAu'(L)=0]{u(0)=0}u(x)=\frac{q_0}{2EA}x(2L-x)$$

(apply BCs *a priori*)

$$u(x)=a_0+a_1x+a_2x^2\xrightarrow[F(L)=EAu'(L)=0]{u(0)=0}u(x)=a_2x(x-2L)$$

$$J=\int_0^L\left[EA(2a_2)+q_0\right]\left(b_0+b_1x+b_2x^2\right)dx=0\rightarrow a_2=-\frac{q_0}{2EA}$$

Example (2)

$\left[\text{method 2}\right]$ balanced-derivative

$$J=\int_0^L\left[EAu''(x)v(x)+q_0v(x)\right]dx=\left[EAu'(x)v(x)\right]_0^L-\int_0^L EAu'(x)v'(x)\,dx+\int_0^L q_0v(x)\,dx=0$$

(i) same smoothness requirements for assumed u and v

(ii) BC appear explicitly in the non-integral term

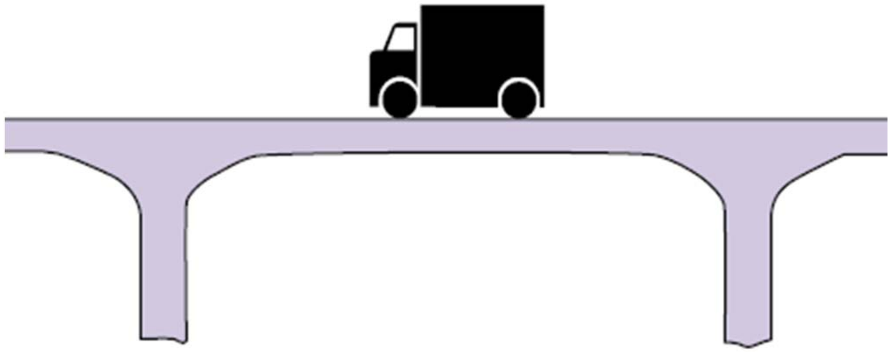
$$\xrightarrow[\delta u(x)\text{ strongly satisfies all essential BC}]{v(x)=\delta u(x)} J=\int_0^L EAu'(x)\delta u'(x)\,dx-\int_0^L q_0\delta u(x)\,dx-\left[EAu'(x)\delta u(x)\right]_0^L\equiv\delta\Pi$$

$$\Pi=U-W=\frac{1}{2}\int_0^L u'(x)EAu'(x)\,dx-\int_0^L q_0u(x)\,dx$$

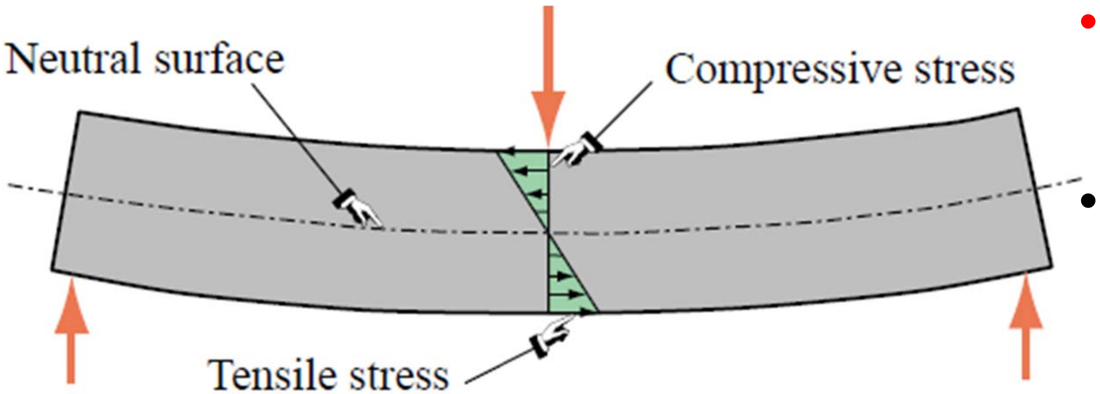
$$J=0\leftrightarrow\delta\Pi=0\leftrightarrow\delta U=\delta W$$

$$\text{Galerkin method}\xleftrightarrow[\text{the Euler-Lagrange equation of a functional}]{\text{if the residual is}}\text{variational formulation}$$

What is a Beam?



Resist primarily transverse loads
General beam > beam-column > beam

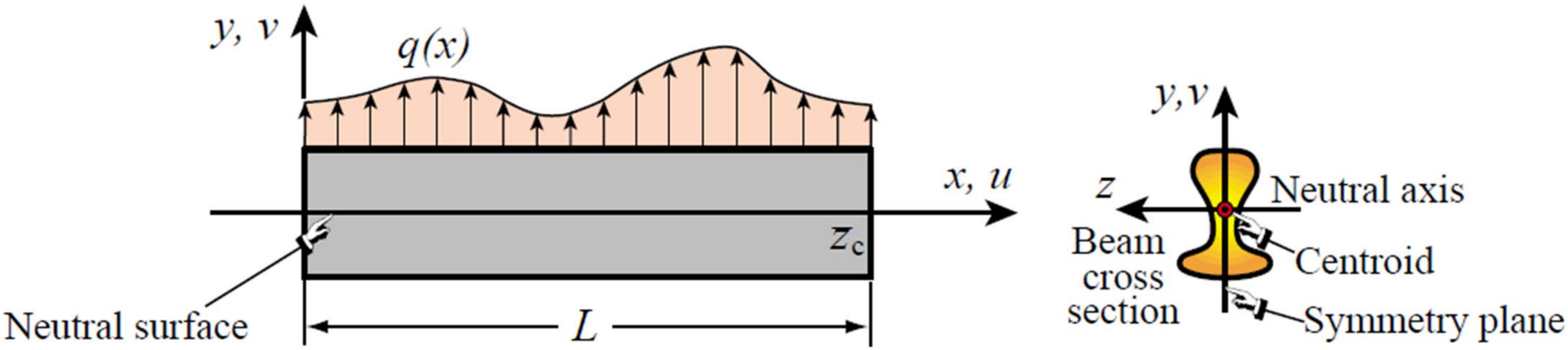


transverse loads → (flexural action) → supports

- Terminology
  - Straight (longitudinal axis)
  - Prismatic (const cross-sec)
- Configuration
  - Spatial
  - Plane
- Model (beam theory)
  - Bernoulli-Euler
    - Hermitian beam element
    - C^1 element
  - Timoshenko
    - C^0 element

Assumptions of Classical Beam Theory

- Planar symmetry
- Cross section variation
- Normality
- Strain energy: only for bending moment deformations
- Linearization
  - So small transverse deflections, rotations and deformations
- Material model: elastic and isotropic

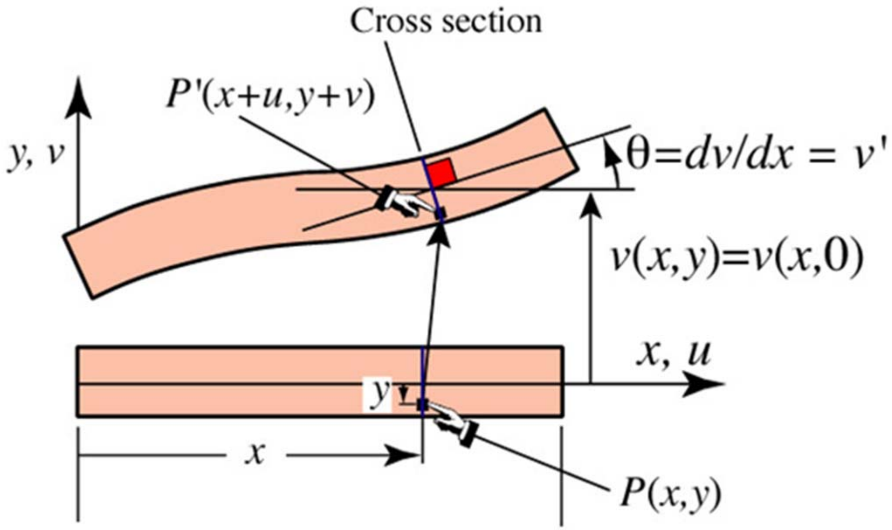


Bernoulli-Euler Beam Theory

[Kinematics]

$$\begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix} = \begin{bmatrix} -y\dfrac{\partial v(x)}{\partial x} \\ v(x) \end{bmatrix} = \begin{bmatrix} -yv' \\ v(x) \end{bmatrix} = \begin{bmatrix} -y\theta \\ v(x) \end{bmatrix}$$

$$\kappa = \frac{d^2v/dx^2}{\left[1+(dv/dx)^2\right]^{3/2}} \approx \frac{\partial^2 v}{\partial x^2}$$

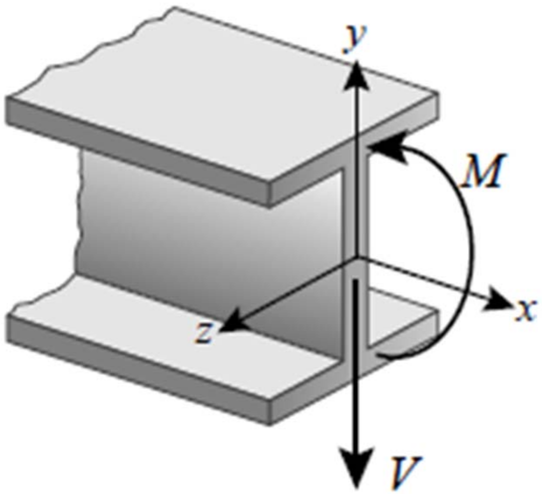


[Strains, Stresses, Bending Moments]

$$e = \frac{\partial u}{\partial x} = -y\frac{\partial^2 v}{\partial x^2} = -yv'' = -y\kappa$$

$$\sigma = Ee = -Ey\frac{\partial^2 v}{\partial x^2} = -Ey\kappa$$

$$M = \int_A -y\sigma dA = E\frac{\partial^2 v}{\partial x^2}\int_A y^2 dA = EI\kappa$$

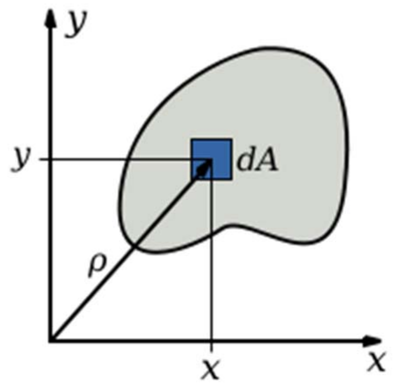


Moment of Inertia

- Mass moment of inertia (관성모멘트)

$$I = kmr^2 = \sum_{i=1}^{n} m_i r_i^2 = \int r^2 dm = \iiint_V r^2 \rho(r) dV \rightarrow I = I_{cm} + md^2$$

- Area moment of inertia
  - Second moment of area (단면이차모멘트): bending
  - Polar moment of inertia (극관성모멘트): torsion
  - Product of inertia: unsymmetric geometry



$$I_{xx} = \int_A y^2 dA \rightarrow I_{xx} = I_{xx\_c} + \bar{x}^2 A \text{ where } \bar{x}A = \int_A x dA$$

$$I_{yy} = \int_A x^2 dA$$

$$J(= I_z) = \int_A \rho^2 dA = \int_A \left(x^2 + y^2\right) dA = \int_A x^2 dA + \int_A y^2 dA = I_{xx} + I_{yy}$$

$$I_{xy} = \int_A xy dA$$

Curvature

– Rate of change of the slope angle of the curve w.r.t. distance along the curve

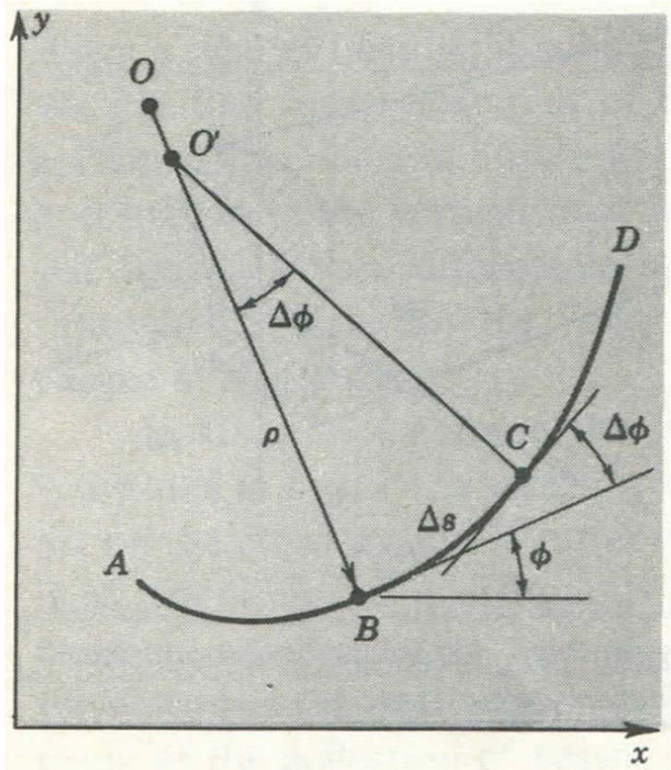
$$\frac{d\phi}{ds} = \lim_{\Delta s \to 0} \frac{\Delta\phi}{\Delta s} == \lim_{\Delta s \to 0} \frac{1}{O'B} = \frac{1}{\rho}$$

ρ : radius of curvature @B

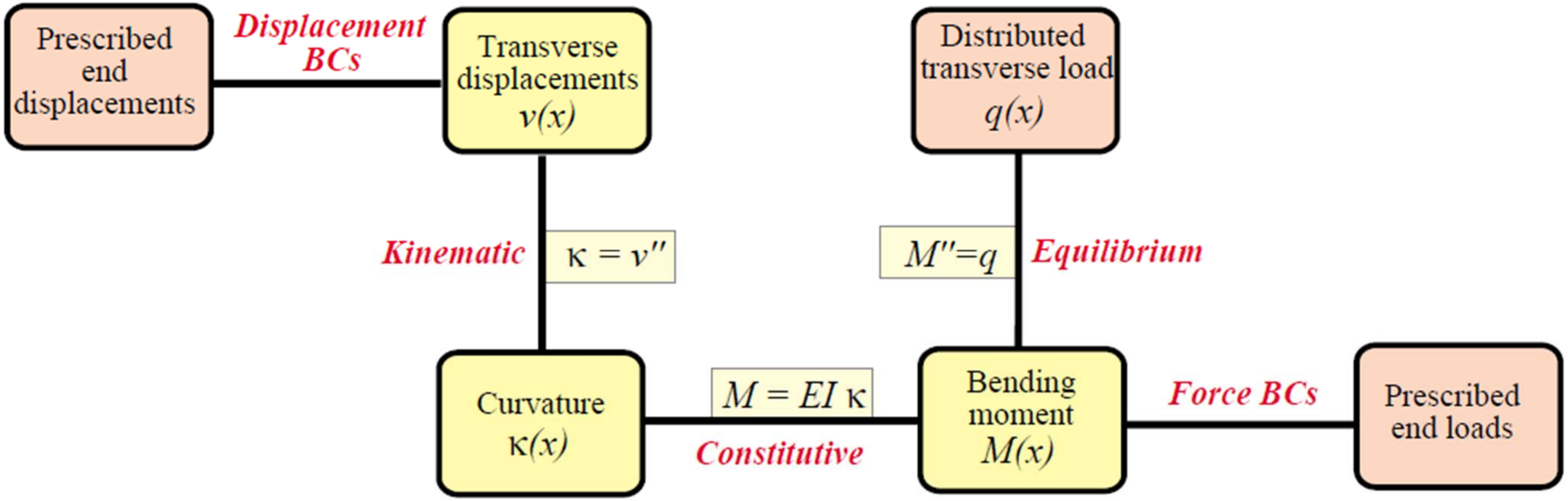
$$\frac{dy}{dx} = \tan\phi \rightarrow \frac{d^2y}{dx^2}\underbrace{\frac{dx}{ds}}_{\cos\phi} = \sec^2\phi \frac{d\phi}{ds}$$

$$\cos\phi = \frac{dx}{ds} = \frac{dx}{\sqrt{dx^2+dy^2}} = \frac{1}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}$$

$$\frac{d\phi}{ds} = \frac{d^2y}{dx^2}\cos^3\phi = \frac{d^2y}{dx^2}\Bigg/\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}$$



Tonti Diagram of the Bernoulli-Euler beam model



[Internal energy due to bending]

$$U = \frac{1}{2}\int_V \sigma e dV = \frac{1}{2}\int_V (-Ey\kappa)(-y\kappa)dV = \frac{1}{2}\int_0^L E\kappa^2 dx \int_A y^2 dA = \frac{1}{2}\int_0^L EI\kappa^2 dx$$

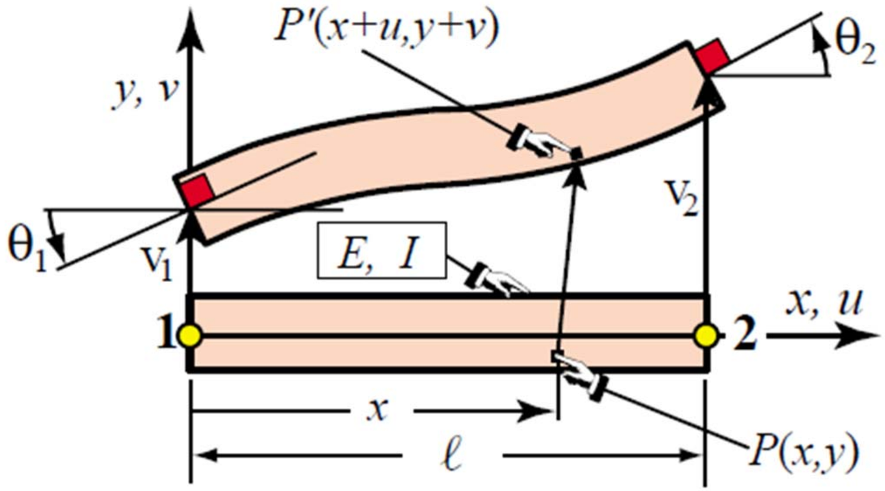
$$= \frac{1}{2}\int_0^L M\kappa dx = \frac{1}{2}\int_0^L EI(v'')^2 dx = \frac{1}{2}\int_0^L v''EIv''dx$$

[External energy due to transverse load q] $W = \int_0^L qvdx$

$$\Pi = U - W$$

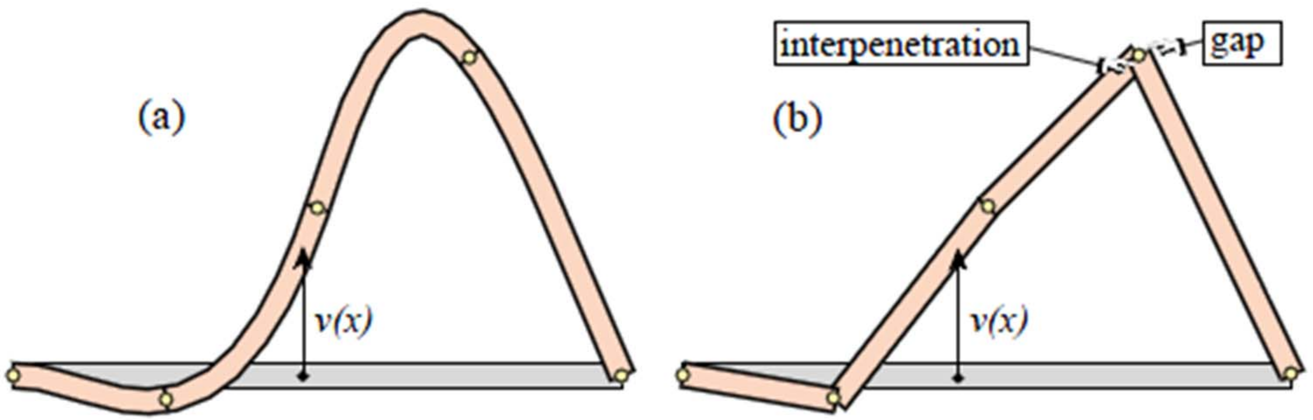
Beam Finite Elements

$$\mathbf{u}^e = \begin{bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{bmatrix}$$



C^1 continuity requirement:

$v(x)$ and $\theta = v'(x) = \dfrac{dv(x)}{dx}$ must be continuous over the entire member and between elements



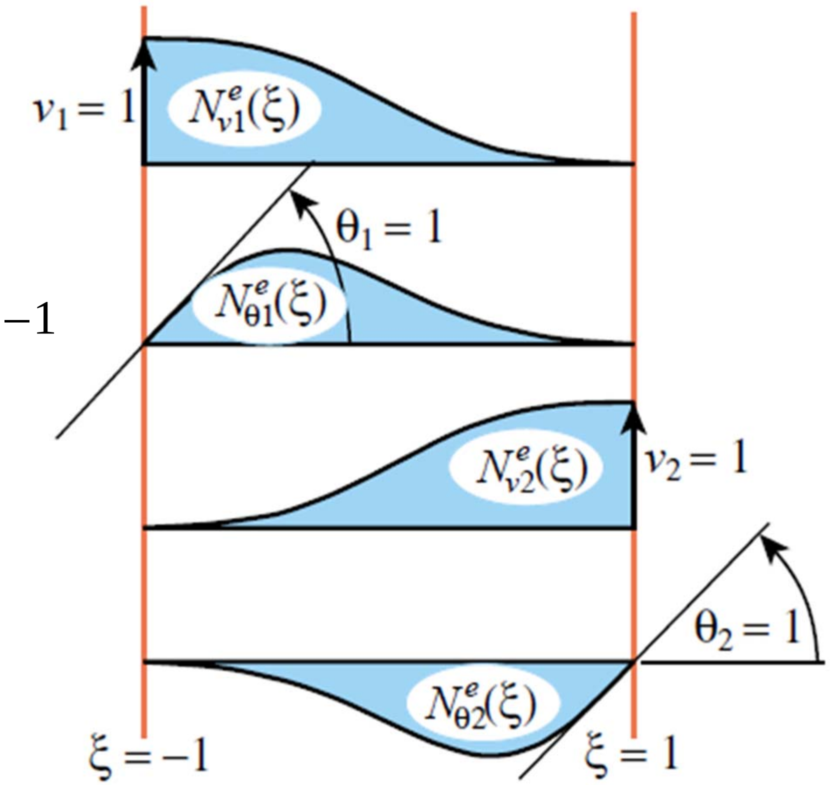
Hermitian Cubic Shape Functions (1)

$$v^e = \begin{bmatrix} N^e_{v_1} & N^e_{\theta_1} & N^e_{v_2} & N^e_{\theta_2} \end{bmatrix} \begin{bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{bmatrix} = \mathbf{N}^e \mathbf{u}^e$$

introduce the natural (isoparametric) coordinate

$$\left.\begin{matrix} x : 0 \sim l \\ \xi : -1 \sim +1 \end{matrix}\right\} \rightarrow \left\{\begin{matrix} \xi(x) = ax + b \\ \xi(0) = -1 \text{ and } \xi(l) = +1 \end{matrix}\right\} \rightarrow \xi = \frac{2x}{l} - 1$$

	$N^e(-1)$	$\dfrac{dN^e}{dx}(-1)$	$N^e(+1)$	$\dfrac{dN^e}{dx}(+1)$
$N^e_{v_1}$	1	0	0	0
$N^e_{\theta_1}$	0	1	0	0
$N^e_{v_2}$	0	0	1	0
$N^e_{\theta_2}$	0	0	0	1



Hermitian Cubic Shape Functions (2)

$$N_{v_1}^e = a_0 + a_1\xi + a_2\xi^2 + a_3\xi^3 \rightarrow \left(N_{v_1}^e\right)' = a_1 + 2a_2\xi + 3a_3\xi^2$$

$$\left.\begin{array}{l} \left.\begin{array}{l} N_{v_1}^e(-1) = a_0 - a_1 + a_2 - a_3 = 1 \\ N_{v_1}^e(+1) = a_0 + a_1 + a_2 + a_3 = 0 \end{array}\right\} 2a_1 + 2a_3 = -1 \\ \left.\begin{array}{l} \left(N_{v_1}^e\right)'(-1) = a_1 - 2a_2 + 3a_3 = 0 \\ \left(N_{v_1}^e\right)'(+1) = a_1 + 2a_2 + 3a_3 = 0 \end{array}\right\} 2a_1 + 6a_3 = 0 \end{array}\right\} a_3 = \frac{1}{4}, a_1 = -\frac{3}{4} \rightarrow \begin{cases} a_0 + a_2 = \frac{1}{2} \\ a_2 = 0 \end{cases}$$

$$N_{v_1}^e = \frac{1}{4}\left(2 - 3\xi + \xi^3\right) = \frac{1}{4}(1-\xi)^2(2+\xi)$$

$$N_{\theta_1}^e = \frac{1}{8} l (1-\xi)^2 (1+\xi)$$

$$N_{v_2}^e = \frac{1}{4}(1+\xi)^2(2-\xi)$$

$$N_{\theta_2}^e = -\frac{1}{8} l (1+\xi)^2 (1-\xi)$$

Curvatures from Displacements

$$\kappa = \frac{d^2 v(x)}{dx^2} = \frac{d^2\mathbf{N}^e}{dx^2}\mathbf{u}^e + \mathbf{N}^e\frac{d^2\mathbf{u}^e}{dx^2} = \frac{d^2\mathbf{N}^e}{dx^2}\mathbf{u}^e = \begin{bmatrix} \frac{d^2N^e_{v_1}}{dx^2} & \frac{d^2N^e_{\theta_1}}{dx^2} & \frac{d^2N^e_{v_2}}{dx^2} & \frac{d^2N^e_{\theta_2}}{dx^2}\end{bmatrix}\begin{bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{bmatrix} = \mathbf{B}\mathbf{u}^e$$

$$\mathbf{B} = \frac{1}{l}\begin{bmatrix} 6\frac{\xi}{l} & 3\xi - 1 & -6\frac{\xi}{l} & 3\xi + 1 \end{bmatrix}$$

$$\frac{df(x)}{dx} = \frac{df(\xi)}{d\xi}\frac{d\xi}{dx} = \frac{df(\xi)}{d\xi}\frac{2}{l} \rightarrow \frac{d^2 f(x)}{dx^2} = \frac{d}{dx}\left(\frac{df(\xi)}{d\xi}\right)\frac{2}{l} + \frac{df(\xi)}{d\xi}\frac{d}{dx}\left(\frac{2}{l}\right) = \frac{4}{l^2}\frac{d^2 f(\xi)}{d\xi^2}$$

$$N^e_{v_1} = \frac{1}{4}(1-\xi)^2(2+\xi) \rightarrow \frac{d^2N^e_{v_1}}{dx^2} = \frac{4}{l^2}\left[\frac{1}{4}\frac{d}{d\xi^2}\left(-2(1-\xi)(2+\xi) + (1-\xi)^2\right)\right] = \frac{4}{l^2}\left(\frac{1}{4}6\xi\right)$$

$$N^e_{\theta_1} = \frac{1}{8}l(1-\xi)^2(1+\xi) \rightarrow \frac{d^2N^e_{\theta_1}}{dx^2} = \frac{4}{l^2}\left[\frac{l}{8}\frac{d}{d\xi^2}\left(-2(1-\xi)(1+\xi) + (1-\xi)^2\right)\right] = \frac{4}{l^2}\left(\frac{l}{8}(6\xi - 2)\right)$$

$$N^e_{v_2} = \frac{1}{4}(1+\xi)^2(2-\xi) \rightarrow \frac{d^2N^e_{v_2}}{dx^2} = \frac{4}{l^2}\left[\frac{1}{4}\frac{d}{d\xi^2}\left(2(1+\xi)(2-\xi) - (1+\xi)^2\right)\right] = \frac{4}{l^2}\left(\frac{1}{4}(-6\xi)\right)$$

$$N^e_{\theta_2} = -\frac{1}{8}l(1+\xi)^2(1-\xi) \rightarrow \frac{d^2N^e_{\theta_2}}{dx^2} = \frac{4}{l^2}\left[-\frac{l}{8}\frac{d}{d\xi^2}\left(2(1+\xi)(1-\xi) - (1+\xi)^2\right)\right] = \frac{4}{l^2}\left(-\frac{l}{8}(-6\xi - 2)\right)$$

Element Stiffness and Consistent Nodal Forces

$$\mathbf{N} = \left[\frac{1}{4}(1-\xi)^2(2+\xi) \quad \frac{1}{8}l(1-\xi)^2(1+\xi) \quad \frac{1}{4}(1+\xi)^2(2-\xi) \quad -\frac{1}{8}l(1+\xi)^2(1-\xi) \right]$$

$$\mathbf{B} = \frac{1}{l}\left[6\frac{\xi}{l} \quad 3\xi - 1 \quad -6\frac{\xi}{l} \quad 3\xi + 1 \right]$$

$$\mathbf{v}^e = \mathbf{N}\mathbf{u}^e \rightarrow \mathbf{v}'' = \mathbf{N}''\mathbf{u}^e = \mathbf{B}\mathbf{u}^e$$

[Internal energy due to bending]

$$U^e = \frac{1}{2}\int_0^L v'' EI v'' dx = \frac{1}{2}\int_0^L \left(\mathbf{u}^e\right)^T \mathbf{B}^T EI \mathbf{B}\mathbf{u}^e dx = \frac{1}{2}\left(\mathbf{u}^e\right)^T \left[\int_0^L \mathbf{B}^T EI \mathbf{B} dx\right]\mathbf{u}^e$$

[External energy due to transverse load q] $W^e = \int_0^L qv dx = \int_0^L q\mathbf{N}^T\mathbf{u}^e dx = \left(\mathbf{u}^e\right)^T \int_0^L q\mathbf{N}^T dx$

$$\Pi^e = U^e - W^e = \frac{1}{2}\left(\mathbf{u}^e\right)^T \underbrace{\left[\int_0^L \mathbf{B}^T EI \mathbf{B} dx\right]}_{\mathbf{K}^e}\mathbf{u}^e - \left(\mathbf{u}^e\right)^T \underbrace{\int_0^L q\mathbf{N}^T dx}_{\mathbf{f}^e}$$

$$\mathbf{K}^e = \int_0^L EI\mathbf{B}^T\mathbf{B} dx = \int_{-1}^{+1} EI\mathbf{B}^T\mathbf{B}\frac{1}{2} l d\xi$$

$$\mathbf{f}^e = \int_0^L q\mathbf{N}^T dx = \int_{-1}^{+1} q\mathbf{N}^T \frac{1}{2} l d\xi$$

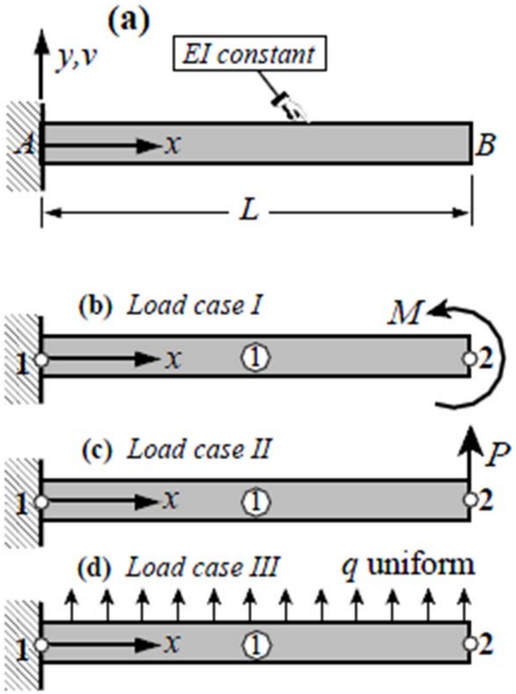
Prismatic Beam and Uniform Load

$$\mathbf{K}^e = \int_{-1}^{+1} EI\mathbf{B}^T\mathbf{B}\frac{1}{2}l\,d\xi = \frac{1}{2}EIl\int_{-1}^{+1}\mathbf{B}^T\mathbf{B}\,d\xi = \frac{1}{2}EIl\int_{-1}^{+1}\frac{1}{l}\begin{bmatrix} 6\frac{\xi}{l} \\ 3\xi-1 \\ -6\frac{\xi}{l} \\ 3\xi+1 \end{bmatrix}\frac{1}{l}\begin{bmatrix} 6\frac{\xi}{l} & 3\xi-1 & -6\frac{\xi}{l} & 3\xi+1 \end{bmatrix}d\xi$$

$$= \frac{1}{2l}EI\int_{-1}^{+1}\begin{bmatrix} 36\xi^2 & 6\xi(3\xi-1)l & -36\xi^2 & 6\xi(3\xi+1)l \\ & (3\xi-1)^2 l^2 & -6\xi(3\xi-1)l & (9\xi^2-1)l^2 \\ & & 36\xi^2 & -6\xi(3\xi+1)l \\ sym & & & (3\xi+1)^2 l^2 \end{bmatrix}d\xi = \frac{EI}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ & 4l^2 & -6l & 2l^2 \\ & & 12 & -6l \\ sym & & & 4l^2 \end{bmatrix}$$

$$\mathbf{f}^e = \int_{-1}^{+1} q\mathbf{N}^T\frac{1}{2}l\,d\xi = \frac{1}{2}ql\int_{-1}^{+1}\mathbf{N}^T d\xi = \frac{1}{2}ql\int_{-1}^{+1}\begin{bmatrix} \frac{1}{4}(1-\xi)^2(2+\xi) \\ \frac{1}{8}l(1-\xi)^2(1+\xi) \\ \frac{1}{4}(1+\xi)^2(2-\xi) \\ -\frac{1}{8}l(1+\xi)^2(1-\xi) \end{bmatrix}d\xi = \frac{1}{2}ql\begin{bmatrix} 1 \\ \frac{1}{6}l \\ 1 \\ -\frac{1}{6}l \end{bmatrix} \rightarrow \begin{cases} \text{two transverse nodal loads: } \frac{1}{2}ql \\ \text{two nodal moments: } \pm\frac{1}{12}ql^2 \end{cases}$$

Example 1: Cantilever Beam



$$\frac{EI}{L^3}\begin{bmatrix}12 & 6L & -12 & 6L\\ 6L & 4L^2 & -6L & 2L^2\\ -12 & -6L & 12 & -6L\\ 6L & 2L^2 & -6L & 4L^2\end{bmatrix}\begin{bmatrix}v_1\\ \theta_1\\ v_2\\ \theta_2\end{bmatrix}=\begin{bmatrix}0\\0\\0\\M\end{bmatrix}\rightarrow\begin{bmatrix}v_2\\ \theta_2\end{bmatrix}=\begin{bmatrix}\frac{ML^2}{2EI}\\ \frac{ML}{EI}\end{bmatrix}$$

$$\frac{EI}{L^3}\begin{bmatrix}12 & 6L & -12 & 6L\\ 6L & 4L^2 & -6L & 2L^2\\ -12 & -6L & 12 & -6L\\ 6L & 2L^2 & -6L & 4L^2\end{bmatrix}\begin{bmatrix}v_1\\ \theta_1\\ v_2\\ \theta_2\end{bmatrix}=\begin{bmatrix}0\\0\\P\\0\end{bmatrix}\rightarrow\begin{bmatrix}v_2\\ \theta_2\end{bmatrix}=\begin{bmatrix}\frac{PL^3}{3EI}\\ \frac{PL^2}{2EI}\end{bmatrix}$$

$$\frac{EI}{L^3}\begin{bmatrix}12 & 6L & -12 & 6L\\ 6L & 4L^2 & -6L & 2L^2\\ -12 & -6L & 12 & -6L\\ 6L & 2L^2 & -6L & 4L^2\end{bmatrix}\begin{bmatrix}v_1\\ \theta_1\\ v_2\\ \theta_2\end{bmatrix}=\frac{1}{2}qL\begin{bmatrix}1\\ \frac{1}{6}\beta L\\ 1\\ -\frac{1}{6}\beta L\end{bmatrix}\rightarrow\begin{bmatrix}v_2\\ \theta_2\end{bmatrix}=\begin{bmatrix}\frac{qL^4(4-\beta)}{24EI}\\ \frac{qL^3(3-\beta)}{12EI}\end{bmatrix}$$

$$\begin{cases}\beta=1: & \text{energy consistent load lumping}\\ \beta=0: & \text{EbE (here same as NbN) load lumping}\end{cases}$$

Load case I: $v=\frac{Mx^2}{2EI},\ \theta=\frac{Mx}{EI}$

Load case II: $\begin{cases}v=\frac{Px^2}{6EI}(3L-x)\\ \theta=\frac{Px}{2EI}(2L-x)\end{cases}$

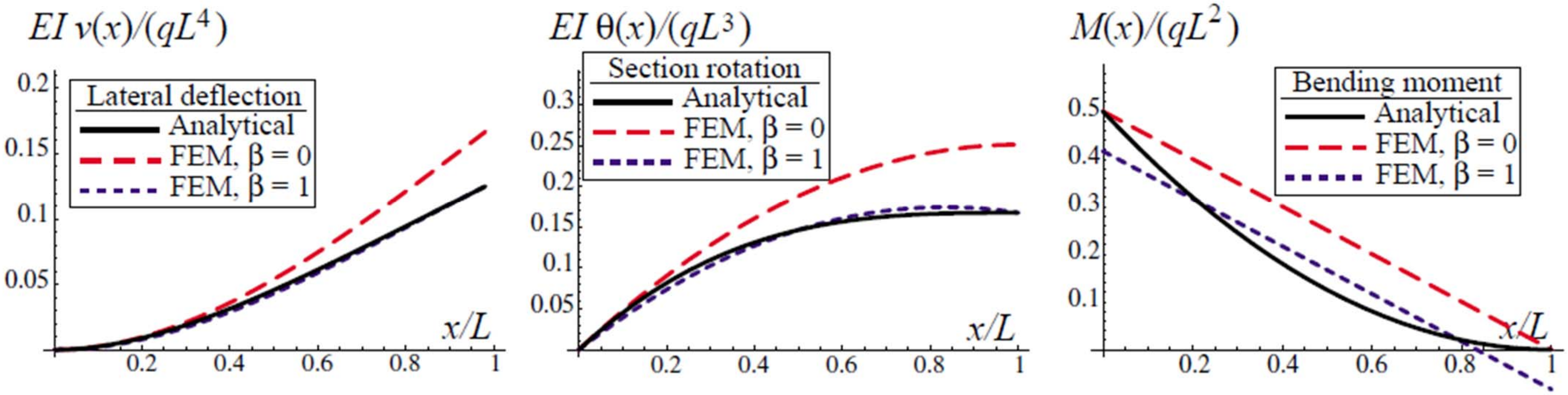
Example 1: Lode case III

[Analytic Solution] $v = \frac{qx^2}{24EI}\left(x^2 + 6l^2 - 4lx\right),\ \theta = \frac{qx}{6EI}\left(x^2 + 3l^2 - 3lx\right),\ M = \frac{q}{2}\left(x^2 - l^2 - 2lx\right)$

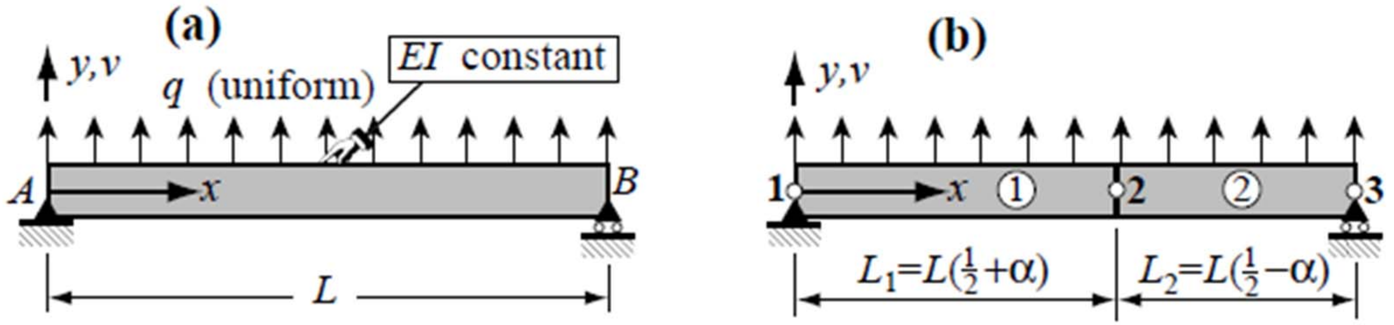
[FEM]

$$\mathbf{v}^e = \mathbf{N}\mathbf{u}^e \to v = \begin{bmatrix} (1-\xi)^2(2+\xi)/4 \\ L(1-\xi)^2(1+\xi)/8 \\ (1+\xi)^2(2-\xi)/4 \\ -L(1+\xi)^2(1-\xi)/8 \end{bmatrix}^T \begin{Bmatrix} 0 \\ 0 \\ v_2 \\ \theta_2 \end{Bmatrix} = \frac{1}{4}\left(\frac{2x}{L}\right)^2\left(3-\frac{2x}{L}\right)\frac{qL^4(4-\beta)}{24EI} - \frac{1}{8}\left(\frac{2x}{L}\right)^2\left(2-\frac{2x}{L}\right)\frac{qL^3(3-\beta)}{12EI} = qL^2x^2\frac{L(6-\beta)-2x}{24EI}$$

$$\theta = \frac{dv}{dx} = qL^2x\frac{L(6-\beta)-3x}{12EI},\ M = EI\frac{d^2v}{dx^2} = \frac{qL}{12}\left[L(6-\beta)-6x\right]$$



Example 2: SS Beam



$$\frac{EI}{L^3}\begin{bmatrix} \frac{12}{(1/2+\alpha)^3} & \frac{6L}{(1/2+\alpha)^2} & \frac{-12}{(1/2+\alpha)^3} & \frac{6L}{(1/2+\alpha)^2} & 0 & 0 \\ \frac{6L}{(1/2+\alpha)^2} & \frac{4L^2}{1/2+\alpha} & -\frac{6L}{(1/2+\alpha)^2} & \frac{2L^2}{1/2+\alpha} & 0 & 0 \\ \frac{-12}{(1/2+\alpha)^3} & -\frac{6L}{(1/2+\alpha)^2} & \frac{12}{(1/2+\alpha)^3}+\frac{12}{(1/2-\alpha)^3} & -\frac{6L}{(1/2+\alpha)^2}+\frac{6L}{(1/2-\alpha)^2} & \frac{-12}{(1/2-\alpha)^3} & \frac{6L}{(1/2-\alpha)^2} \\ \frac{6L}{(1/2+\alpha)^2} & \frac{2L^2}{1/2+\alpha} & -\frac{6L}{(1/2+\alpha)^2}+\frac{6L}{(1/2-\alpha)^2} & \frac{4L^2}{1/2+\alpha}+\frac{4L^2}{1/2-\alpha} & \frac{-6L}{(1/2-\alpha)^2} & \frac{2L^2}{1/2-\alpha} \\ 0 & 0 & \frac{-12}{(1/2-\alpha)^3} & \frac{-6L}{(1/2-\alpha)^2} & \frac{12}{(1/2-\alpha)^3} & \frac{-6L}{(1/2-\alpha)^2} \\ 0 & 0 & \frac{6L}{(1/2-\alpha)^2} & \frac{2L^2}{1/2-\alpha} & \frac{-6L}{(1/2-\alpha)^2} & \frac{4L^2}{1/2-\alpha} \end{bmatrix}\begin{bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \\ v_3 \\ \theta_3 \end{bmatrix} = \frac{1}{2}qL\begin{bmatrix} 1/2+\alpha \\ \frac{1}{6}L(1/2+\alpha)^2 \\ 1/2+\alpha+1/2-\alpha \\ -\frac{1}{6}L(1/2+\alpha)^2+\frac{1}{6}L(1/2-\alpha)^2 \\ 1/2-\alpha \\ -\frac{1}{6}L(1/2-\alpha)^2 \end{bmatrix}$$

$$\rightarrow v_2 = \frac{qL^4\left(5-24\alpha^2+16\alpha^4\right)}{384EI}$$

$$v(x) = \frac{qL^4\left(\zeta-2\zeta^3+\zeta^4\right)}{24EI} \quad \text{where} \quad \zeta = \frac{x}{L} \xrightarrow{x=L_1=L\left(\frac{1}{2}+\alpha\right)} v_2^{\text{exact}} = \frac{qL^4\left(5-24\alpha^2+16\alpha^4\right)}{384EI}$$

(v and θ inside elements will NOT agree with the exact one.)

Accuracy Analysis: Nodal Exactness

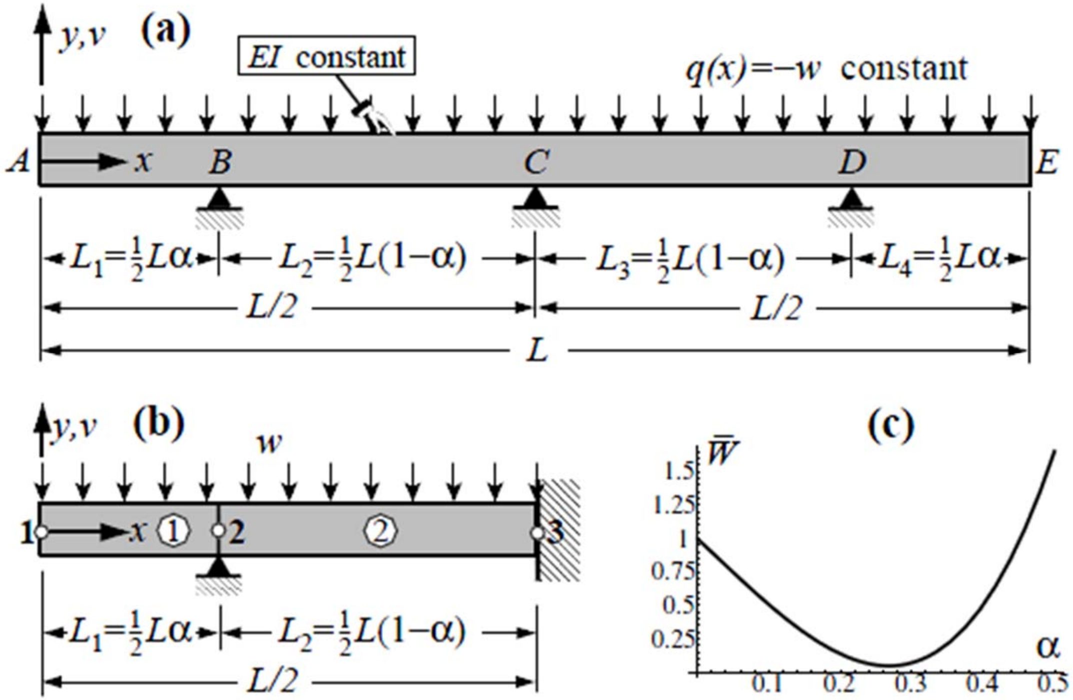
- Low order 1D elements (two-node bar) can display infinite accuracy under some conditions
  - The bar properties are constant along the length (prismatic member)
  - The distributed load *q(x)* is zero between nodes. The only applied loads are point forces at the nodes.
- linear axial displacement u(x) is the exact solution over each element since constant strain and stress satisfy, element by element, all of the governing equations
  - truss discretizations: one element per member is enough if they are prismatic and loads are applied to joints

Accuracy Analysis: Superconvergence

- What happens if the foregoing assumptions are not met?
  - Exactness is then generally lost, and several elements per member may be beneficial if spurious mechanisms are avoided
- For a 1D lattice of equal-length, prismatic two-node bar elements, an interesting and more difficult result is:
  - *the solution is nodally exact for any loading if consistent node forces are used*
- If conditions such as equal-length are relaxed, the solution is no longer nodally exact but convergence at the nodes is extremely rapid (faster than could be expected by standard error analysis) as long as consistent node forces are used

Example 3: Continuum Beam

Optimal location of supports ?



best α ? →

- Minimum external energy: $W(\alpha) = \mathbf{f}^T \mathbf{u} \rightarrow dW/d\alpha = 0 \rightarrow \alpha \approx 0.27$
- Equal reactions: $R_B = R_C, (R_D = R_B) \rightarrow f_2^r = 2 f_3^r \rightarrow \alpha = 0.30546 (R_B = R_C = R_D = wL/3)$
- Minimum relative deflection: $v_{ji}^{\max}(\alpha) = \max |v_j - v_i| \rightarrow \alpha = 0.26681 \left(v_{ji}^{\max} \leq wL^4 / (67674 EI) \right)$
- Minimum absolute moment: $M^{\max}(\alpha) = \max |M(x, \alpha)| \rightarrow \alpha = 0.25540 \left(M^{\max} \leq wL^2 / 589 \right)$

Example 3: Continuum Beam

$$\frac{4EI}{L^3}\begin{bmatrix} \frac{24}{\alpha^3} & \frac{6L}{\alpha^2} & -\frac{24}{\alpha^3} & \frac{6L}{\alpha^2} & 0 & 0 \\ \frac{6L}{\alpha^2} & \frac{2L^2}{\alpha} & -\frac{6L}{\alpha^2} & \frac{L^2}{\alpha} & 0 & 0 \\ -\frac{24}{\alpha^3} & -\frac{6L}{\alpha^2} & \frac{24\left(1-3\alpha\hat{\alpha}\right)}{\alpha^3\hat{\alpha}^3} & -\frac{6L(1-2\alpha)}{\alpha^2\hat{\alpha}^2} & -\frac{24}{\hat{\alpha}^3} & \frac{6L}{\hat{\alpha}^2} \\ \frac{6L}{\alpha^2} & \frac{L^2}{\alpha} & -\frac{6L(1-2\alpha)}{\alpha^2\hat{\alpha}^2} & \frac{2L^2}{\alpha\hat{\alpha}} & -\frac{6L}{\hat{\alpha}^2} & \frac{L^2}{\hat{\alpha}} \\ 0 & 0 & -\frac{24}{\hat{\alpha}^3} & -\frac{6L}{\hat{\alpha}^2} & \frac{24}{\hat{\alpha}^3} & -\frac{6L}{\hat{\alpha}^2} \\ 0 & 0 & \frac{6L}{\hat{\alpha}^2} & \frac{L^2}{\hat{\alpha}} & -\frac{6L}{\hat{\alpha}^2} & \frac{2L^2}{\hat{\alpha}} \end{bmatrix}\begin{bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \\ v_3 \\ \theta_3 \end{bmatrix} = \frac{wL}{4}\begin{bmatrix} -\alpha \\ \frac{-L\alpha^2}{12} \\ -1 \\ \frac{L(2\alpha-1)}{12} \\ -\hat{\alpha} \\ \frac{L\hat{\alpha}^2}{12} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ f_2^r \\ 0 \\ f_3^r \\ m_3^r \end{bmatrix}$$

$$\frac{4EI}{L^3}\begin{bmatrix} \frac{24}{\alpha^3} & \frac{6L}{\alpha^2} & \frac{6L}{\alpha^2} \\ \frac{6L}{\alpha^2} & \frac{2L^2}{\alpha} & \frac{L^2}{\alpha} \\ \frac{6L}{\alpha^2} & \frac{L^2}{\alpha} & \frac{2L^2}{\alpha\hat{\alpha}} \end{bmatrix}\begin{bmatrix} v_1 \\ \theta_1 \\ \theta_2 \end{bmatrix} = \frac{wL}{4}\begin{bmatrix} -\alpha \\ \frac{-L\alpha^2}{12} \\ \frac{L(2\alpha-1)}{12} \end{bmatrix}$$

$$v_1 = -\frac{wL^4}{768\,EI}\,\alpha\left((1+\alpha)^3-2\right), \quad \theta_1 = \frac{wL^3}{384\,EI}\left((1+\alpha)^3-2\right), \quad \theta_2 = \frac{wL^3}{384\,EI}\,\hat{\alpha}\,(1-2\alpha-5\alpha^2)$$

$$f_{r2} = \frac{wL}{16}\,\frac{3+2\alpha+\alpha^2}{\hat{\alpha}}, \quad f_{r3} = \frac{wL}{16}\,\frac{5-10\alpha-\alpha^2}{\hat{\alpha}}, \quad m_{r3} = -\frac{wL^2}{32}(1-2\alpha-\alpha^2)$$