EXERCISE 16.5

(a) From the definition of element geometry in the second row of (E16.3):

$$x = x_1 N_1^e + x_2 N_2^e + x_3 N_3^e = \ell\left[\tfrac{1}{2}\xi(\xi+1) + \left(\tfrac{1}{2}+\alpha\right)(1-\xi^2)\right], \qquad \text{(E16.12)}$$

we get

$$J = \frac{\partial x}{\partial \xi} = \left(\tfrac{1}{2} - 2\xi\alpha\right)\ell. \qquad \text{(E16.13)}$$

(In 1D, the Jacobian is a scalar and is the same as its determinant.) J is linear in ξ. Consequently its max/min values over the element occur at the end nodes:

$$J_1 = J|_{\xi=-1} = (\tfrac{1}{2} - 2\alpha)\ell, \quad J_2 = J|_{\xi=1} = (\tfrac{1}{2} + 2\alpha)\ell \qquad \text{(E16.14)}$$

Both J_1 and J_2 are obviously positive if $-\frac{1}{4} < \alpha < \frac{1}{4}$, and if so $J > 0$ inside the element. If $\alpha = 0$, $J = \frac{1}{2}\ell$ is constant over the element.

(b) The strain displacement matrix linking $e = \mathbf{B}\,\mathbf{u}^e$ is given by

$$\mathrm{B} = \frac{d\mathbf{N}^e}{dx} = \frac{d\mathbf{N}^e}{d\xi}\frac{d\xi}{dx} = J^{-1}\left[\frac{\partial N_1^e}{\partial \xi}\ \frac{\partial N_2^e}{\partial \xi}\ \frac{\partial N_e^e}{\partial \xi}\right] = \frac{1}{\ell(\frac{1}{2} - 2\alpha\xi)}\left[\xi - \tfrac{1}{2} \quad \xi + \tfrac{1}{2} \quad -2\xi\right] \qquad \text{(E16.15)}$$

(c) From Chapter 12 the element strain energy is (12.18). Replacing the $e = \mathbf{B}\,\mathbf{u}^e$ of item (b) gives $U^e = \frac{1}{2}(\mathbf{u}^e)^T \int_0^\ell \mathbf{B}^T\,\mathbf{E}\,\mathbf{B}\,dx\;\mathbf{u}^e \overset{\text{def}}{=} \frac{1}{2}(\mathbf{u}^e)^T\mathbf{K}^e\,\mathbf{u}^e$, whence

$$\mathbf{K}^e = \int_0^\ell \mathbf{B}^T\mathbf{E}\mathbf{B}\,dx = \int_{-1}^{1} \mathbf{B}^T\mathbf{E}\,\mathbf{B}\,\frac{dx}{d\xi}\,d\xi = \int_{-1}^{1} \mathbf{B}^T\mathbf{E}\,\mathbf{B}\,J\,d\xi. \qquad \text{(E16.16)}$$

For the two-point Gauss rule the *Mathematica* script of Figure E16.4 gives

$$\mathbf{K}^e = \frac{EA}{\ell(3-16\alpha^2)}\begin{bmatrix} 7-16\alpha & 1 & -8(1-2\alpha) \\ 1 & 7+16\alpha & -8(1+2\alpha) \\ -8(1-2\alpha) & -8(1+2\alpha) & 16 \end{bmatrix} \qquad \text{(E16.17)}$$

If $\alpha = 0$ this reduces to

$$\mathbf{K}^e = \frac{EA}{3\ell}\begin{bmatrix} 7 & 1 & -8 \\ 1 & 7 & -8 \\ -8 & -8 & 16 \end{bmatrix} \qquad \text{(E16.18)}$$

(d) If $\alpha = 0$, from item (a) $J = \frac{1}{2}\ell$ is constant whereas $\mathbf{B}$ is linear in ξ. Because EA is constant, $EA\,\mathbf{B}^T\,\mathbf{B}\,J$ is a *quadratic* polynomial in ξ. This is exactly integrated by a Gauss rule with 2 or more points.

EXERCISE 16.7 The *Mathematica* script of Figure E16.7 gives the complicated result:

$$\mathbf{f}^e = \frac{qL}{24}\begin{bmatrix} 3\xi_L^2 - 2(1+4\alpha)\xi_L^3 + 6\alpha\xi_L^4 + \xi_R^2(-3+2(1+\alpha(4-3\xi_R))\xi_R) \\ -3\xi_L^2 + (-2+8\alpha)\xi_L^3 + 6\alpha\xi_L^4 + \xi_R^2(3-2\xi_R(-1+\alpha(4+3\xi_R))) \\ -4(\xi_L(3-\xi_L^2+3\alpha\xi_L(-2+\xi_L^2)) - 3\xi_R + 6\alpha\xi_R^2 + \xi_R^3 - 3\alpha\xi_R^4) \end{bmatrix} \tag{E16.19}$$

If $\alpha = 0$, $\xi_L = -1$ and $\xi_R = 1$ so the load extends over the whole bar, this expression simplifies to

$$\mathbf{f}^e = \frac{qL}{6}[1 \quad 1 \quad 4]^T. \tag{E16.20}$$

This is nothing but Simpson's rule for integration.

EXERCISE 17.1 This exercise can be done using the *Mathematica* script listed in Figure E17.5. Running that code produces the results shown in Figure E17.6.

```
ClearAll[Em,ν,a,b,h];  Em=48; h=1; a=4; b=2; ν=0;
ncoor={{0,0},{a,0},{a,b},{0,b}};
Emat=Em/(1-ν^2)*{{1,ν,0},{ν,1,0},{0,0,(1-ν)/2}};
For [p=1, p<=4, p++,
      Ke= Quad4IsoPMembraneStiffness[ncoor,Emat,h,{True,p}];
      Print["Gauss integration rule: ",p," x ",p];
      Print["Ke=",Chop[Ke]//MatrixForm];
      Print["Eigenvalues of Ke=",Chop[Eigenvalues[N[Ke]]]]
   ];
```

FIGURE E17.5. Script to do Exercise 17.1.

Gauss integration rule: 1 x 1

$$\text{Ke} = \begin{pmatrix} 18. & 6. & 6. & -6. & -18. & -6. & -6. & 6. \\ 6. & 27. & 6. & 21. & -6. & -27. & -6. & -21. \\ 6. & 6. & 18. & -6. & -6. & -6. & -18. & 6. \\ -6. & 21. & -6. & 27. & 6. & -21. & 6. & -27. \\ -18. & -6. & -6. & 6. & 18. & 6. & 6. & -6. \\ -6. & -27. & -6. & -21. & 6. & 27. & 6. & 21. \\ -6. & -6. & -18. & 6. & 6. & 6. & 18. & -6. \\ 6. & -21. & 6. & -27. & -6. & 21. & -6. & 27. \end{pmatrix}$$

Eigenvalues of Ke= {896., 60., 24., 0, 0, 0, 0, 0}

Gauss integration rule: 2 x 2

$$\begin{pmatrix} 24. & 6. & 0 & -6. & -12. & -6. & -12. & 6. \\ 6. & 36. & 6. & 12. & -6. & -18. & -6. & -30. \\ 0 & 6. & 24. & -6. & -12. & -6. & -12. & 6. \\ -6. & 12. & -6. & 36. & 6. & -30. & 6. & -18. \\ -12. & -6. & -12. & 6. & 24. & 6. & 0 & -6. \\ -6. & -18. & -6. & -30. & 6. & 36. & 6. & 12. \\ -12. & -6. & -12. & 6. & 0 & 6. & 24. & -6. \\ 6. & -30. & 6. & -18. & -6. & 12. & -6. & 36. \end{pmatrix}$$

Eigenvalues of Ke= {896., 60., 36., 24., 24., 0, 0, 0}

Gauss integration rule: 3 x 3

$$\text{Ke} = \begin{pmatrix} 24. & 6. & 0 & -6. & -12. & -6. & -12. & 6. \\ 6. & 36. & 6. & 12. & -6. & -18. & -6. & -30. \\ 0 & 6. & 24. & -6. & -12. & -6. & -12. & 6. \\ -6. & 12. & -6. & 36. & 6. & -30. & 6. & -18. \\ -12. & -6. & -12. & 6. & 24. & 6. & 0 & -6. \\ -6. & -18. & -6. & -30. & 6. & 36. & 6. & 12. \\ -12. & -6. & -12. & 6. & 0 & 6. & 24. & -6. \\ 6. & -30. & 6. & -18. & -6. & 12. & -6. & 36. \end{pmatrix}$$

Eigenvalues of Ke= {896., 60., 36., 24., 24., 0, 0, 0}

Gauss integration rule: 4 x 4

$$\text{Ke} = \begin{pmatrix} 24. & 6. & 0 & -6. & -12. & -6. & -12. & 6. \\ 6. & 36. & 6. & 12. & -6. & -18. & -6. & -30. \\ 0 & 6. & 24. & -6. & -12. & -6. & -12. & 6. \\ -6. & 12. & -6. & 36. & 6. & -30. & 6. & -18. \\ -12. & -6. & -12. & 6. & 24. & 6. & 0 & -6. \\ -6. & -18. & -6. & -30. & 6. & 36. & 6. & 12. \\ -12. & -6. & -12. & 6. & 0 & 6. & 24. & -6. \\ 6. & -30. & 6. & -18. & -6. & 12. & -6. & 36. \end{pmatrix}$$

Eigenvalues of Ke= {896., 60., 36., 24., 24., 0, 0, 0}

FIGURE E17.6. Results from running the script of Figure E17.5.

The $\mathbf{K}^e$ given by the 1×1 Gauss rule has a rank deficiency of two because it has five zero eigenvalues instead of three. This behavior is explained in Chapter 19.

The 2×2, 3×3 and 4×4 rules produce the same stiffness matrix. This matrix has three zero eigenvalues, which correspond to the three independent rigid body modes in two dimensions.

The reason for the repeating stiffness matrices is that the integrand $h\mathbf{B}^T\mathbf{E}\mathbf{B}\,J$ is at most quadratic in ξ and η because h and $\mathbf{E}$ are constant, $\mathbf{B}$ is linear in ξ and η, and *for a rectangle* J is constant over the element. A 2-point product Gauss rule is exact for up to cubic polynomials in the ξ and η directions, so it does quadratics exactly.