



FIGURE E18.6. Lines used for direct construction of shape functions of the 10-node cubic triangle.

EXERCISE 18.1

- (a) The shape functions for nodes 1, 4 and 0 of the 10-node cubic triangle can be obtained using the lines marked in red in Figure E18.6. The process is as follows.

$$N_1^e(\zeta_1, \zeta_2, \zeta_3) = c_1 \zeta_1 (\zeta_1 - \frac{1}{3}) (\zeta_1 - \frac{2}{3})$$

$$N_1^e(1, 0, 0) = c_1 \times 1 \times \frac{2}{3} \times \frac{1}{3} = \frac{2}{9} c_1 = 1, \quad \text{hence } c_1 = \frac{9}{2}, \quad (\text{E18.4})$$

$$N_1^e = \frac{9}{2} \zeta_1 (\zeta_1 - \frac{1}{3}) (\zeta_1 - \frac{2}{3}) = \frac{1}{2} \zeta_1 (3\zeta_1 - 1)(3\zeta_1 - 2)$$

$$N_4^e(\zeta_1, \zeta_2, \zeta_3) = c_4 \zeta_1 \zeta_2 (\zeta_1 - \frac{1}{3})$$

$$N_4^e(\frac{2}{3}, \frac{1}{3}, 0) = c_4 \times \frac{2}{3} \times \frac{1}{3} \times \frac{1}{3} = \frac{2}{27} c_4 = 1, \quad \text{hence } c_4 = \frac{27}{2}, \quad (\text{E18.5})$$

$$N_4^e = \frac{27}{2} \zeta_1 \zeta_2 (\zeta_1 - \frac{1}{3}) = \frac{9}{2} \zeta_1 \zeta_2 (3\zeta_1 - 1)$$

$$N_0^e(\zeta_1, \zeta_2, \zeta_3) = c_0 \zeta_1 \zeta_2 \zeta_3$$

$$N_0^e(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}) = c_0 \times \frac{1}{3} \times \frac{1}{3} \times \frac{1}{3} = \frac{1}{27} c_0 = 1, \quad \text{hence } c_0 = 27, \quad (\text{E18.6})$$

$$N_0^e = 27 \zeta_1 \zeta_2 \zeta_3.$$

The remaining shape functions can be obtained by appropriate index substitutions:

$$N_2^e = \frac{9}{2} \zeta_2 (\zeta_2 - \frac{1}{3}) (\zeta_2 - \frac{2}{3}), \quad N_3^e = \frac{9}{2} \zeta_3 (\zeta_3 - \frac{1}{3}) (\zeta_3 - \frac{2}{3}),$$

$$N_4^e = \frac{27}{2} \zeta_1 \zeta_2 (\zeta_1 - \frac{1}{3}), \quad N_5^e = \frac{27}{2} \zeta_1 \zeta_2 (\zeta_2 - \frac{1}{3}),$$

$$N_6^e = \frac{27}{2} \zeta_2 \zeta_3 (\zeta_2 - \frac{1}{3}), \quad N_7^e = \frac{27}{2} \zeta_2 \zeta_3 (\zeta_3 - \frac{1}{3}),$$

$$N_8^e = \frac{27}{2} \zeta_3 \zeta_1 (\zeta_3 - \frac{1}{3}), \quad N_9^e = \frac{27}{2} \zeta_3 \zeta_1 (\zeta_1 - \frac{1}{3}). \quad (\text{E18.7})$$

Here factors $9/2$ and $27/2$ may be partially absorbed to in the factors to make $\zeta_1 - \frac{1}{3} \rightarrow 3\zeta_1 - 1$, etc.

- (b) The verification that the sum is unity is done with the *Mathematica* script shown in Figure E18.7. Notice the substitution enforcing $\zeta_1 + \zeta_2 + \zeta_3 = 1$.

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Nf={ζ1*(3*ζ1-1)*(3*ζ1-2)/2, ζ2*(3*ζ2-1)*(3*ζ2-2)/2,
  ζ3*(3*ζ3-1)*(3*ζ3-2)/2, 9*ζ1*ζ2*(3*ζ1-1)/2, 9*ζ1*ζ2*(3*ζ2-1)/2,
  9*ζ2*ζ3*(3*ζ2-1)/2, 9*ζ2*ζ3*(3*ζ3-1)/2, 9*ζ3*ζ1*(3*ζ3-1)/2,
  9*ζ3*ζ1*(3*ζ1-1)/2, 27*ζ1*ζ2*ζ3};
S=Simplify[Sum[Nf[[i]],{i,10}]/.{ζ3->1-ζ1-ζ2}]; Print[S];
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FIGURE E18.7. Unit sum verification for 10-node cubic triangle.

EXERCISE 18.6 The shape function for the internal node 5 is the bubble function $N_5 = (1 - \xi^2)(1 - \eta^2)$, which also appears in Eq. (15.17) as N_9 of the 9-node quad. Take $N_i = \bar{N}_i + c_5 N_5$, where $\bar{N}_i$, $i = 1, 2, 3, 4$ are the shape functions (16.15) of the four-node bilinear quadrilateral. These functions are not zero at node 5, hence the correction by a multiple of the bubble function. Evaluation at node 5 ($\xi = \eta = 0$) shows that $c_5 = -1/4$. The corrected corner shape functions are $N_i = \bar{N}_i - N_5/4$, $i = 1, 2, 3, 4$. For example,

$$N_1(\xi, \eta) = \frac{1}{4}(1 - \xi)(1 - \eta) - \frac{1}{4}(1 - \xi^2)(1 - \eta^2). \quad (\text{E18.10})$$

These four corner shape functions vary linearly over all sides because the bubble function N_5 vanishes over the 4 sides. Hence interelement continuity is maintained.

Unit sum check: $N_1 + N_2 + N_3 + N_4 + N_5 = \bar{N}_1 + \bar{N}_2 + \bar{N}_3 + \bar{N}_4 - 4 \times \frac{1}{4} N_5 + N_5 = \bar{N}_1 + \bar{N}_2 + \bar{N}_3 + \bar{N}_4 = 1$, since it is known that the shape functions $\bar{N}_i$ of the 4-node quad add up to one.