1. (1)

$$\left.\begin{array}{l}\text{design variables:}\begin{cases} b = \text{base of the container} \\ h = \text{height of the container}\end{cases} \\ \text{objective function:}\dfrac{\text{round-trip cost of shipping the container}}{\text{one-way cost of shipping the contents}} = \dfrac{2(18)(80)(2b^2+4bh)}{18(150)(b^2h)} \\ \text{constraints: } 0 \le b \le 10, 0 \le h \le 18 \end{array}\right\}$$

$$\rightarrow \begin{cases} \underset{b,h}{\text{Min}}\ f = \left(\dfrac{32}{15}\right)\left(\dfrac{1}{h}+\dfrac{2}{b}\right) \\ \text{subject to}\begin{cases} g_1 = -b \le 0 \\ g_2 = b-10 \le 0 \\ g_3 = -h \le 0 \\ g_4 = h-18 \le 0 \end{cases} \end{cases}$$ (10 points)

(2)

$$\begin{cases} \underset{x_1,x_2,x_3,x_4}{\text{maximize}} & 2x_1 + x_2 + 7x_3 + 4x_4 \quad (5\text{ pts}) \\ \text{subject to} & x_1 + x_2 + x_3 + x_4 = 26 \quad (3\text{ pts}) \\ & x_1, x_2, x_3, x_4 \ge 0 \quad (2\text{ pts}) \end{cases}$$

(3)

resource 1 per day: $x_A + 0.5x_B$

resource 2 per day: $0.2x_A + 0.5x_B$

total cost of resource 1 and 2 per day

$$C = (x_A + 0.5x_B)[0.375 - 0.00005(x_A + 0.5x_B)] + (0.2x_A + 0.5x_B)[0.75 - 0.0001(0.2x_A + 0.5x_B)]$$

return per day from the sale of products A and B

$$R = x_A(2.00 - 0.0005x_A - 0.00015x_B) + x_B(3.50 - 0.0002x_A - 0.0015x_B)$$

$$\begin{cases} \underset{x_A,x_B}{\text{maximize}} & \text{profit } P = R - C \quad (4\text{ pts}) \\ \text{subject to} & \left.\begin{array}{l} x_A + 0.5x_B \le 1000 \ [\text{requirement of resource 1 per day}] \\ 0.2x_A + 0.5x_B \le 250 \ [\text{requirement of resource 2 per day}] \end{array}\right\} (5\text{ pts}) \\ & x_A, x_B \ge 0 \quad (1\text{ pts}) \end{cases}$$

2.

(1) Minimize $f(x_1, x_2) = x_1^3 - 16x_1 + 2x_2 - 3x_2^2$ subject to $x_1 + x_2 \le 3$

$$L = x_1^3 - 16x_1 + 2x_2 - 3x_2^2 + u\left(x_1 + x_2 - 3 + s^2\right)$$

$$\left.\begin{array}{l} \dfrac{\partial L}{\partial x_1} = 3x_1^2 - 16 + u = 0 \\ \dfrac{\partial L}{\partial x_2} = 2 - 6x_2 + u = 0 \\ \dfrac{\partial L}{\partial u} = x_1 + x_2 - 3 + s^2 = 0 \\ \dfrac{\partial L}{\partial s} = us = 0 \\ u \ge 0 \end{array}\right\} \rightarrow \begin{cases} i)\ u = 0: \begin{cases} x_1 = \dfrac{4}{\sqrt{3}}, x_2 = \dfrac{1}{3}, s^2 > 0, f = -24.30 \\ x_1 = -\dfrac{4}{\sqrt{3}}, x_2 = \dfrac{1}{3}, s^2 > 0, f = -24.97 \end{cases} \\ ii)\ s = 0: \begin{cases} x_1 = 0, x_2 = 3, u = 16, f = -21 \\ x_1 = 2, x_2 = 1, u = 4, f = -25 \end{cases} \end{cases}$$

(10 pts)

(2) regularity check @ x^*

There is only one active constraint $\rightarrow$ regularity is satisfied. (5 pts)

(3) sufficient condition (10 pts)

$$\nabla^2 L = \begin{bmatrix} 6x_1 & 0 \\ 0 & -6 \end{bmatrix}, \nabla g = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

	x_1^*	x_2^*	u^*	f^*	g	$\nabla^2 L$
i)	2.3094	0.3333	0	-24.30	inactive	Indefinite
ii)	-2.3094	0.3333	0	-24.97	inactive	Negative definite
iii)	0	3	16	-21	active	Negative semidefinite
iv)	2	1	4	-25	active	Indefinite

i) Since no constraint is active, this is an inflection point for the cost function which does not satisfy the sufficient condition.

ii) Since no constraint is active, this is local maximum point (satisfy sufficient condition for local maximum.)

iii) This point does not satisfy second order necessary condition. It cannot be a local minimum.

iv) $\nabla g \cdot \boldsymbol{d} = 0 \rightarrow \boldsymbol{d} = c\begin{bmatrix} 1 \\ -1 \end{bmatrix} \rightarrow Q = \boldsymbol{d}^T\left(\nabla^2 L\right)\boldsymbol{d} = c^2\begin{bmatrix} 1 & -1 \end{bmatrix}\begin{bmatrix} 12 & 0 \\ 0 & -6 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = 6c^2 > 0$

This point is an isolated local minimum point.

(4) $\Delta f = -u^* e = -4e$

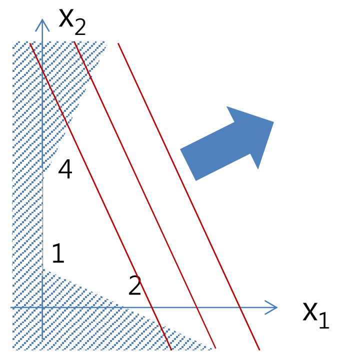
A small positive e results in decrease of cost function.(5 pts)

3.

$$\left.\begin{array}{ll}\text{Maximize} & z = 4x_1 + 2x_2 \\ \text{subject to} & -2x_1 + x_2 \le 4 \\ & x_1 + 2x_2 \ge 2 \\ & x_1,\ x_2 \ge 0\end{array}\right\} \rightarrow \left\{\begin{array}{ll}\min f = -4x_1 - 2x_2 & \\ \text{subject to} & -2x_1 + x_2 + x_3 = 4 \\ & x_1 + 2x_2 - x_4 + x_5 = 2 \\ & x_1,\ x_2 \ge 0 \\ \min w = x_5 = 2 - x_1 - 2x_2 + x_4 & \end{array}\right.$$

(1) unbounded problem

basic	x_1	x_2	x_3	x_4	x_5	b
x_3	-2	1	1	0	0	4
x_5	1	2	0	-1	1	2
cost	-4	-2	0	0	0	f-0
artificial	-1	-2	0	1	0	w-2
x_3	-5/2	0	1	1/2	-1/2	3
x_2	1/2	1	0	-1/2	1/2	1
cost	-3	0	0	-1	1	f+2
artificial	0	0	0	0	1	w-0
x_3	0	5	1	-2	2	8
x_1	1	2	0	-1	1	2
cost	0	6	0	-4	4	f+8



unbounded problem

4. (1)

For $x_1 + 2x_2 \leq 5$: $y_1 = 0$ (c_3' in the slack variable column x_3)
$x_1 + x_2 = 4$: $y_2 = 2.5$ (c_5' in the slack variable column x_5)
$x_1 - x_2 \geq 3$: $y_3 = -1.5$ (c_6' in the artificial variable column x_6)

Therefore, $y_1 = 0.0$, $y_2 = 2.5$, $y_3 = -1.5$

(2)

For $b_1 = 5$: $-\frac{0.5}{1} \leq \Delta_1 \leq \infty$ or $-0.5 \leq \Delta_1 \leq \infty$;
For $b_2 = 4$: $\max\{-\frac{0.5}{0.5}, -\frac{3.5}{0.5}\} \leq \Delta_2 \leq \frac{0.5}{1.5}$ or $-1.0 \leq \Delta_2 \leq 0.333$;
For $b_3 = 3$: $\max\{-\frac{0.5}{0.5}, -\frac{3.5}{0.5}\} \leq \Delta_3 \leq \frac{0.5}{0.5}$ or $-1.0 \leq \Delta_3 \leq 1.0$

(3)

For $c_1 = -1$: $-\frac{1.5}{0.5} \leq \Delta c_1 \leq \infty$ or $-3.0 \leq \Delta c_1 \leq \infty$;
For $c_2 = -4$: $-\infty \leq \Delta c_2 \leq \frac{1.5}{0.5}$ or $-\infty \leq \Delta c_2 \leq 3.0$

For the original form:
For $c_1 = 1$: $-\infty \leq \Delta c_1 \leq 3.0$;
For $c_2 = 4$: $-3.0 \leq \Delta c_2 \leq \infty$