Mixed Variables (MV-OPT)

- Discrete variable
  - Value assigned from a given set of values
  - Plate thickness, material properties, diameter of reinforcing bars
- Integer variable
  - Not divisible as fractions, special class of discrete variables
  - Number of bolts, number of teeth in a gear
- Linked discrete variable
  - Values for a group of parameters
- Binary variable
  - Value of 0 or 1

- Treatment
  - Large values: variable as continuous and rounding off the optimum solution to the nearest integer
  - Small values: ?

Definition of MV-OPT

- extended by defining some of the variables as continuous and others as discrete
- includes integer variable as well as 0–1 variable problems

$$\underset{\mathbf{x}}{\text{Minimize}}\; f\left(\mathbf{x}\right)$$

$$\text{subject to}\; \begin{cases} h_i\left(\mathbf{x}\right)=0; & i=1,\ldots,p \\ g_j\left(\mathbf{x}\right)\le 0; & i=1,\ldots,m \\ x_i \in D_i,\; D_i=\left(d_{i1},d_{i2},\ldots,d_{iq_i}\right); & i=,\ldots,n_d \\ x_{iL}\le x_i \le x_{iU}; & i=\left(n_d+1\right),\ldots,n \end{cases}$$

n_d : number of discrete design variables

D_i : set of allowable discrete values for the *i*th discrete variable

q_i : number of allowable discrete values

d_{ik} : *k*th possible discrete value for the *i*th discrete variable

Classification of MV-OPT (1)

- Assumption: focus only on the discrete variables
  - continuous variables in the problem can be treated with an appropriate continuous variable optimization method
  - if appropriate, a continuous variable is transformed into a discrete variable by defining a grid for it
- MV-OPT 1: Functions Continuous and Differentiable
  - plate thickness from specified values and member radii from the ones available in the market
- MV-OPT 2: Functions Nondifferentiable
  - design problems where constraints from a design code are imposed (experiments and experience)
- MV-OPT 3: Discrete Variables Cannot Have Nondiscrete Values
  - number of strands in a prestressed beam or column, the number of teeth in a gear, and the number of bolts for a joint

Classification of MV-OPT (2)

- MV-OPT 4: Design Variables Linked to Other Parameters
  - structural design with members selected from a catalog, material selection, and engine-type selection for automotive
- MV-OPT 5: Combinatorial Problems
  - traveling salesman problem, design of a bolt insertion sequence, a welding sequence, and a member placement sequence between a given set of nodes

MV-OPT	Variable types	Functions differentiable?	Functions defined at nondiscrete points?	Nondiscrete values allowed for discrete variables?	Variables linked?
1	Mixed	Yes	Yes	Yes	No
2	Mixed	No	Yes	Yes	No
3	Mixed	Yes/No	No	No	No
4	Mixed	Yes/No	No	No	Yes
5	Discrete	No	No	No	Yes/No

Overview of Solution Concepts

- Two basic classes
  - Enumerative: full enumeration($N_c = \prod_{i=1}^{n_d} q_i$) → partial enumeration: branch-and-bound type methods (BBM)
  - Stochastic: simulated annealing, genetic algorithms, and other such algorithms
- at a discrete optimum point, none of the inequalities may be active
- final solution is affected by how widely separated the allowable discrete values are in the sets *Di*
- MV-OPT 1 type
  - solve it first using a continuous variable optimization method
  - optimum value: lower bound for a discrete optimum solution
  - requirement of discreteness of design variables represents additional constraints

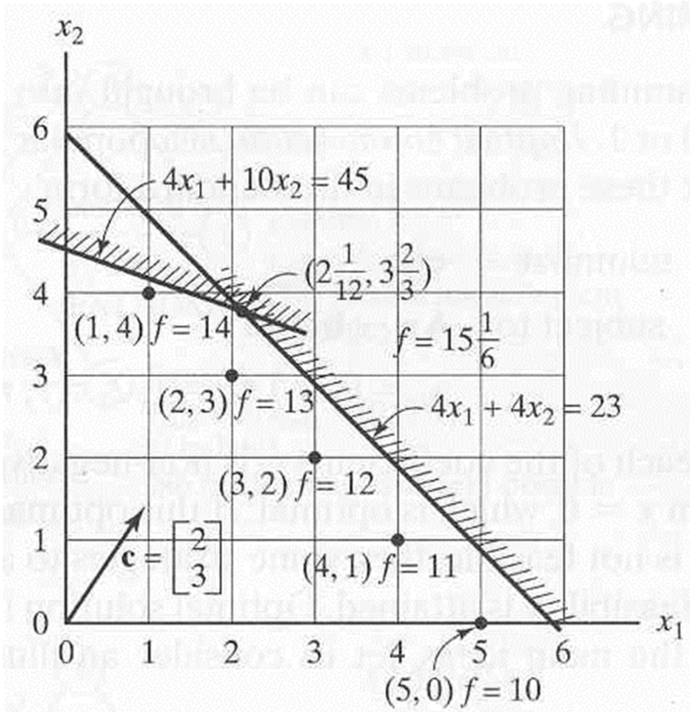
Example

- A furniture manufacturer produces lawn chairs and tables. The profit from the sale of each chair is \$2 and the profit from the sale of a table is \$3. Each chair weighs four pounds and each table weighs ten pounds. A supply of 45 pounds of material is available at hand. The labor per chair is four hours and the labor per table is also four hours. Only 23 hours of labor is available. Determine the number of chairs and tables for realizing maximum profit.

$$
\begin{aligned}
&\textit{Maximize} && f(x_1, x_2) = 2x_1 + 3x_2 \\
&\text{subject to} && 4x_1 + 10x_2 \le 45 \\
& && 4x_1 + 4x_2 \le 23 \\
& && x_1, x_2 \ge 0 \text{ and integer}
\end{aligned}
$$

$$\left[\boldsymbol{x}^* = (2\frac{1}{12}, 3\frac{2}{3}),\ f^* = 15\frac{1}{16} \right]$$

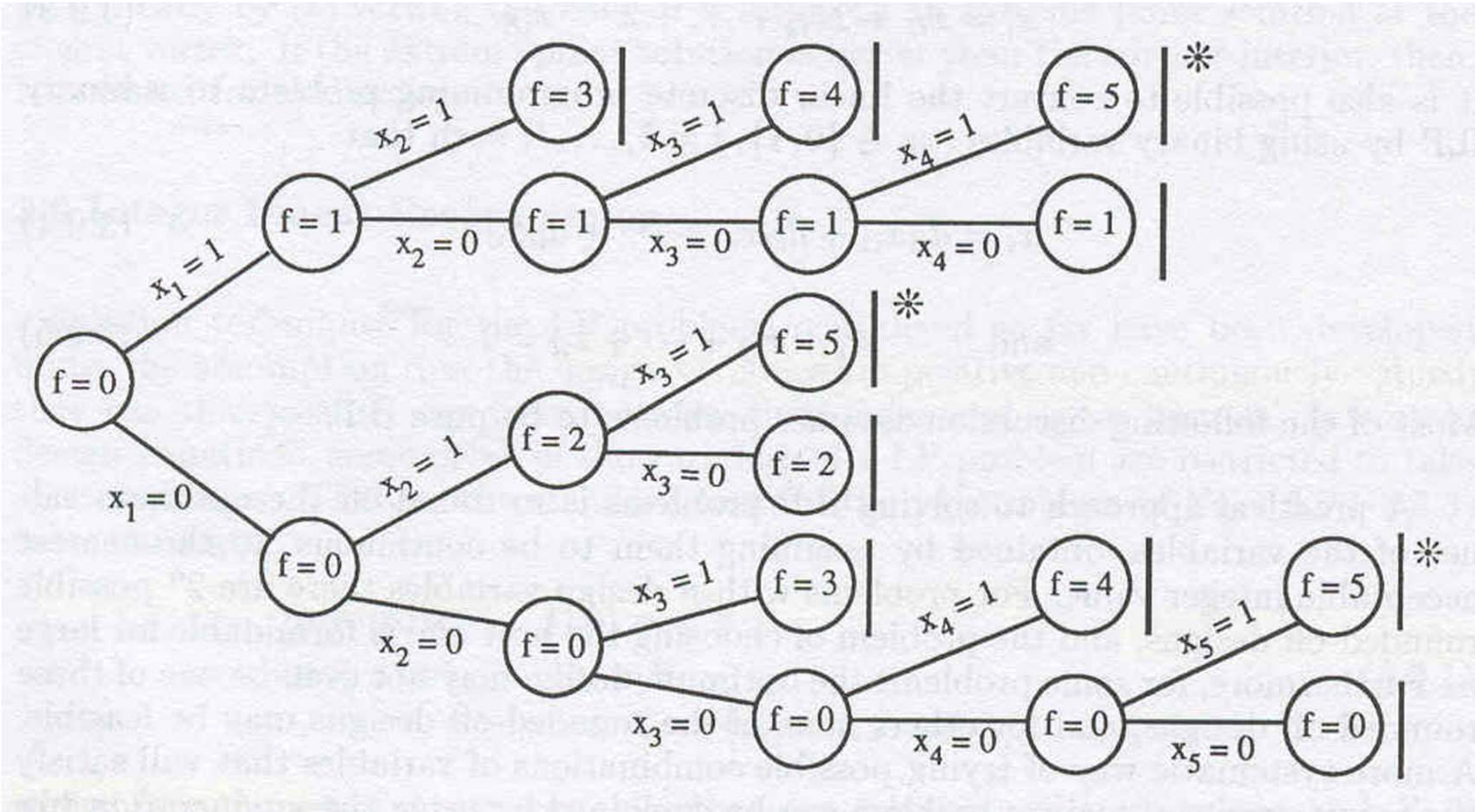
# of chairs	1	2	3	4	5
# of tables	4	3	2	1	0
profit	14	13	12	11	10



Enumeration Tree

$$f = \sum_{i=1}^{5} i x_i = 5 \qquad \left(2^5 \text{ cases} \xrightarrow{\text{40\% reduction}} 19 \text{ cases}\right)$$

$$1x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 = 5$$



Zero-One Programming (1)

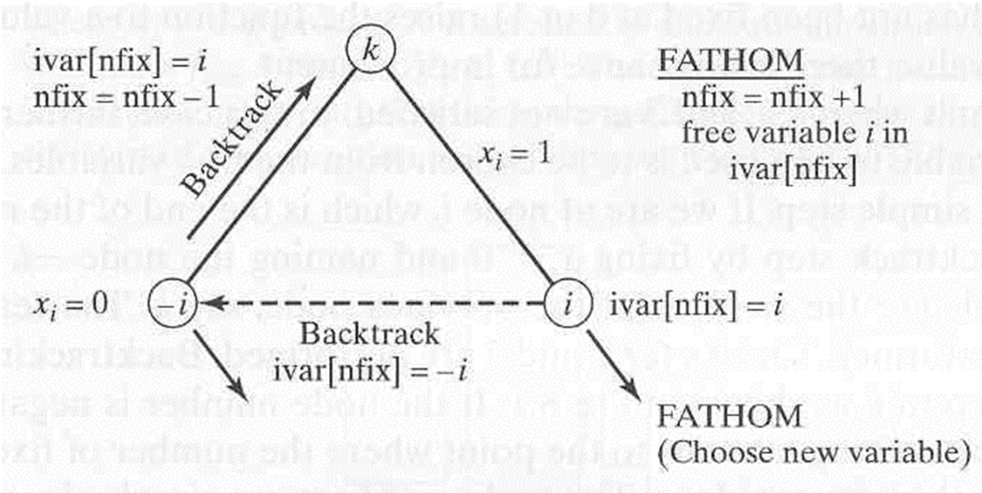
- Variable values: 0 or 1
- Implicit enumeration
  - Standard form

$$\boxed{\begin{array}{ll} \text{Minimize} & \boldsymbol{f} = \boldsymbol{c}^T\boldsymbol{x} \quad (c_i \geq 0) \\ \text{subject to} & \boldsymbol{A}\boldsymbol{x} - \boldsymbol{b} \geq 0 \\ & x_i = 0 \text{ or } 1 \quad (i = 1, \ldots, n) \end{array}}$$

  - Classification
    - Case 1: The point is feasible.
    - Case 2: Feasibility is impossible from this branch.
    - Case 3: The function cannot be improved from a previous feasible value.
    - Case 4: Possibilities for feasibility and improvement exist.

Zero-One Programming (2)

- Case 1~3: branch is fathomed $\rightarrow$ backtrack
- Case 4: further search
- Fathoming
  - systematic procedure to determine whether improvements could be obtained by changing the levels of variables in succession
  - step where a new variable to be brought in is chosen

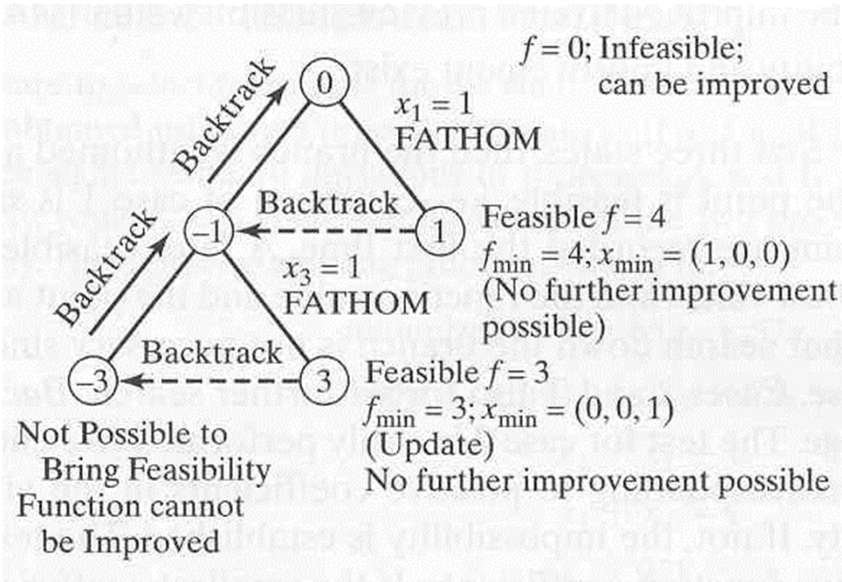


Example

$$\begin{aligned} \text{Minimize} \quad & f = 4x_1 + 5x_2 + 3x_3 \\ \text{subject to} \quad & g_1 = x_1 - 3x_2 - 6x_3 + 6 \geq 0 \\ & g_2 = 2x_1 + 3x_2 + 3x_3 - 2 \geq 0 \\ & g_3 = x_1 + x_3 - 1 \geq 0 \\ & x_i = 0 \text{ or } 1 \end{aligned}$$

(Explicit enumeration: all the possible values are tried)

x_1	x_2	x_3	g_1	g_2	g_3	f
0	0	0	6	–2	– 1	0
0	0	1	0	1	0	3
0	1	0	3	1	– 1	5
0	1	1	– 3	4	0	8
1	0	0	7	0	0	4
1	0	1	1	3	1	7
1	1	0	4	3	0	9
1	1	1	– 2	6	1	12



Branch and Bound (1)

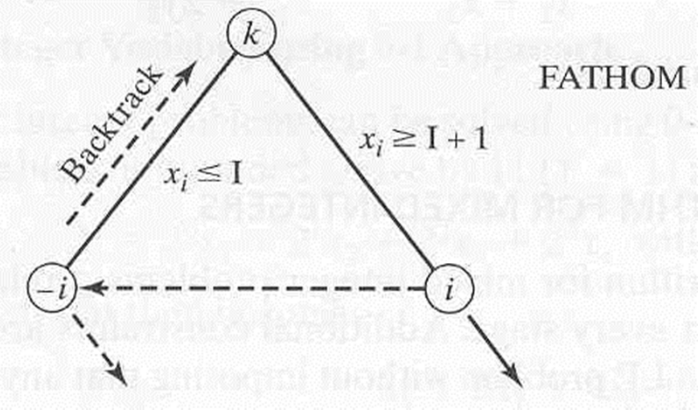
- Land and Doig (1960) $\rightarrow$ Dakin (1965)
- Mixed integers problems
- A relaxed linear programming problem is solved at every stage.
- Additional constraints are then imposed.
  - (1) The problem has a solution, and the required variables are integers.
  - (2) No feasible solution exists.
  - (3) The objective function cannot be improved from its value at a previous feasible integer solution.
  - (4) Some variables have fractional values and the function may be improved.

$$x_i = I + \alpha_i \rightarrow \begin{cases} x_i \geq I + 1 \\ x_i \leq I \end{cases}$$

$I = \lfloor x_i \rfloor$: the largest integer bounded by x_i

α_i : fractional part $\left(0 < \alpha_i < 1\right)$

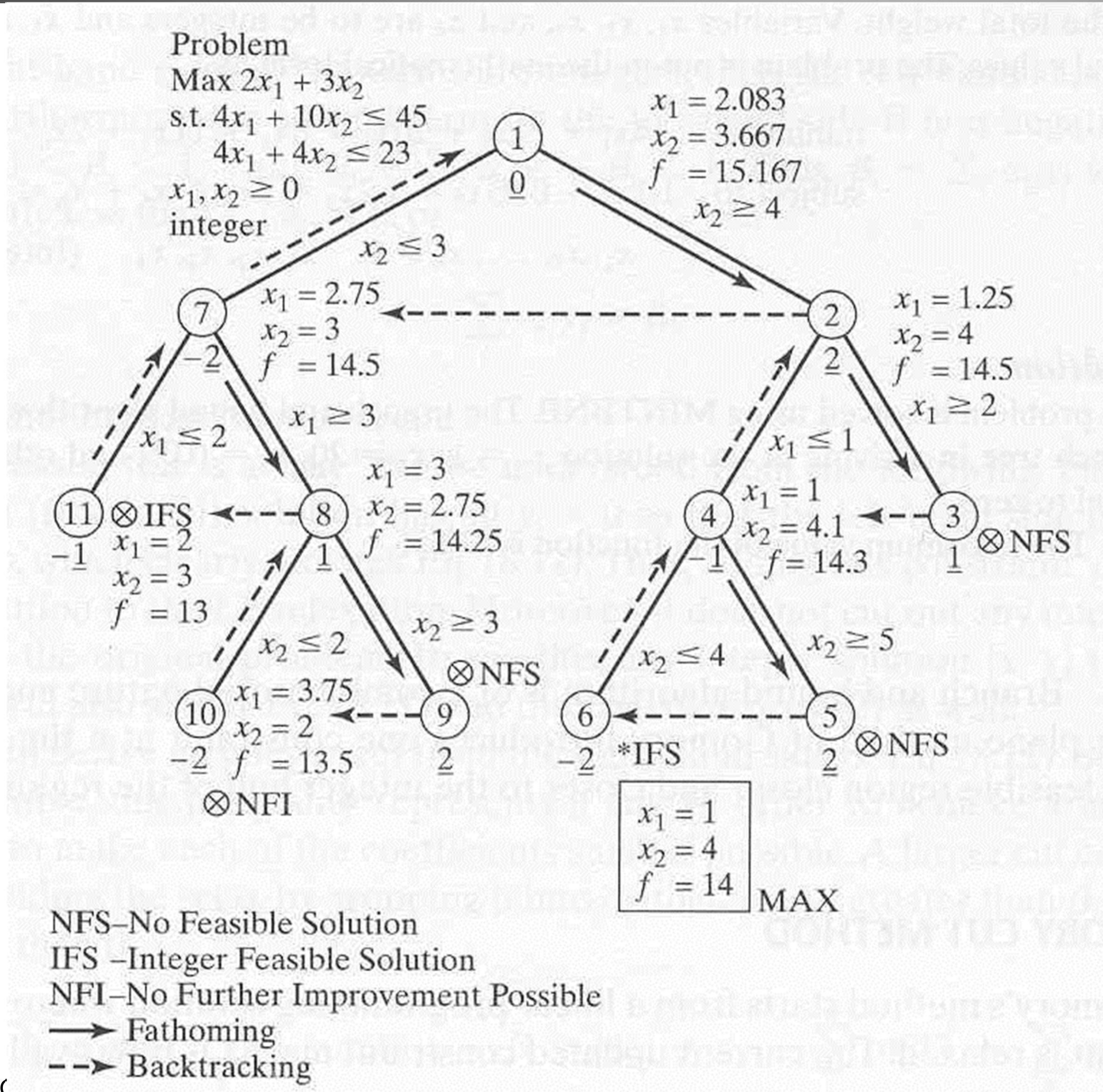
$e.g.,\ 2.3 = 2 + 0.3,\ -2.3 = -3 + 0.7$



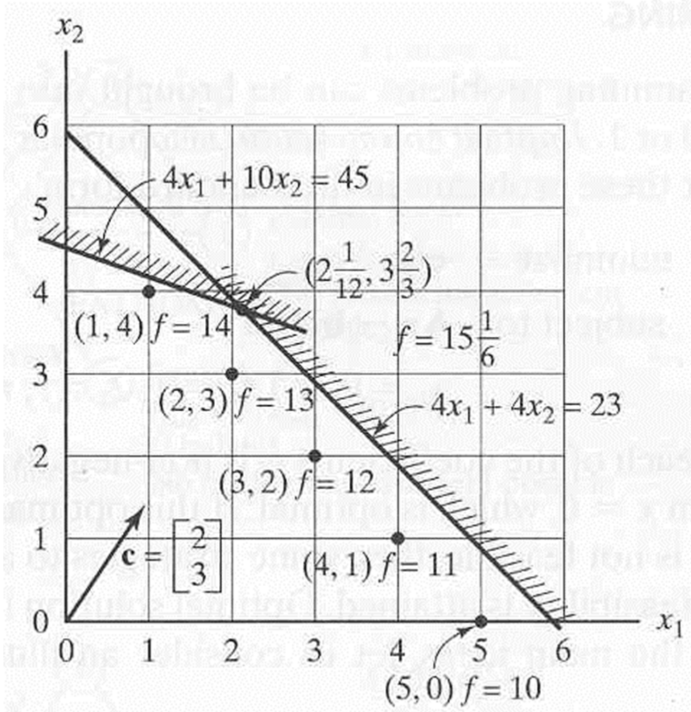
Branch and Bound (2)

- Procedure
  - Solve the MILP by assuming the variables to be continuous
    - If all variables have integer values, stop. Otherwise, set f_L
  - Branch from the node into two new LP problems by adding a new constraint until all the nodes are fathomed
    - No feasible solution
    - All integer feasible solution
    - Other solution ($f < f_U$, noninteger variable)
- Dependence
  - Choice of noninteger variable for branching
    - Variable with the largest fraction
  - Selection of node to be branched
    - Smallest value of the objective function

Example



Problem
Max $2x_1 + 3x_2$
s.t. $4x_1 + 10x_2 \le 45$
$4x_1 + 4x_2 \le 23$
$x_1, x_2 \ge 0$
integer
1
0
$x_1 = 2.083$
$x_2 = 3.667$
$f = 15.167$
$x_2 \ge 4$
$x_2 \le 3$
2
2
$x_1 = 1.25$
$x_2 = 4$
$f = 14.5$
$x_1 \ge 2$
7
−2
$x_1 = 2.75$
$x_2 = 3$
$f = 14.5$
$x_1 \ge 3$
$x_1 \le 2$
$x_1 \le 1$
3
1
⊗NFS
4
−1
$x_1 = 1$
$x_2 = 4.1$
$f = 14.3$
11
−1
⊗IFS
$x_1 = 2$
$x_2 = 3$
$f = 13$
8
1
$x_1 = 3$
$x_2 = 2.75$
$f = 14.25$
$x_2 \ge 3$
$x_2 \le 2$
⊗NFS
$x_2 \le 4$
$x_2 \ge 5$
10
−2
$x_1 = 3.75$
$x_2 = 2$
$f = 13.5$
⊗NFI
9
2
6
−2
*IFS
5
2
⊗NFS
$x_1 = 1$
$x_2 = 4$
$f = 14$
MAX
NFS–No Feasible Solution
IFS –Integer Feasible Solution
NFI–No Further Improvement Possible
Fathoming
Backtracking



x_2
x_1
$4x_1 + 10x_2 = 45$
$(2\frac{1}{12}, 3\frac{2}{3})$
$(1, 4)\ f = 14$
$f = 15\frac{1}{6}$
$(2, 3)\ f = 13$
$4x_1 + 4x_2 = 23$
$(3, 2)\ f = 12$
$\mathbf{c} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$
$(4, 1)\ f = 11$
$(5, 0)\ f = 10$
0
1
2
3
4
5
6

Example 15.1: only discrete values

Minimize $f\left(x_1, x_2\right) = -20x_1 - 10x_2$

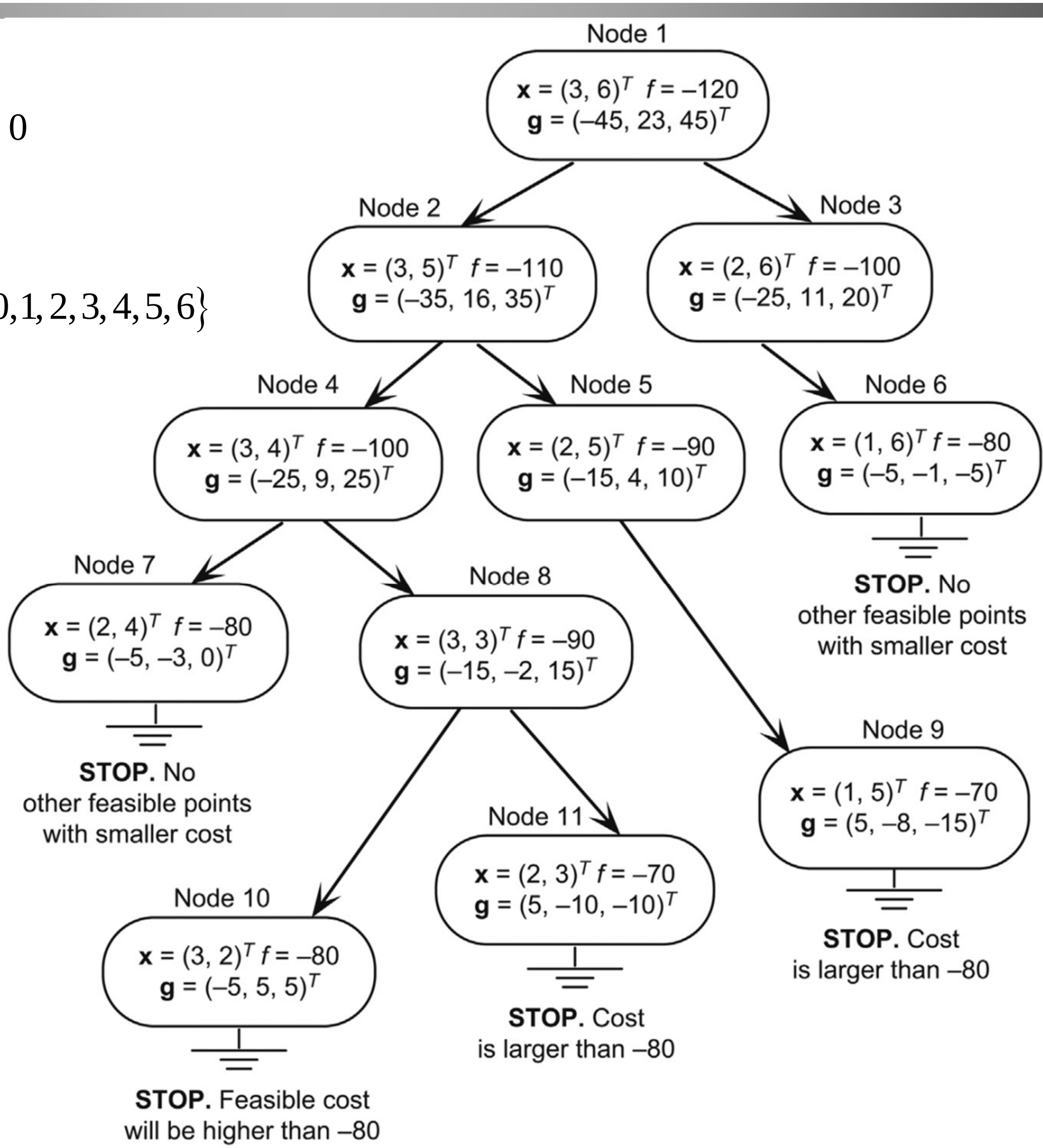
subject to $g_1 = -20x_1 - 10x_2 + 75 \le 0$

$$g_2 = 12x_1 + 7x_2 - 55 \le 0$$

$$g_3 = 25x_1 + 10x_2 - 90 \le 0$$

$$x_1 \in \{0,1,2,3\} \text{ and } x_2 \in \{0,1,2,3,4,5,6\}$$

full enumeration: 28 combinations



Example 15.2: nondiscrete values

$$\text{Minimize} \quad f(x_1, x_2) = -20x_1 - 10x_2$$

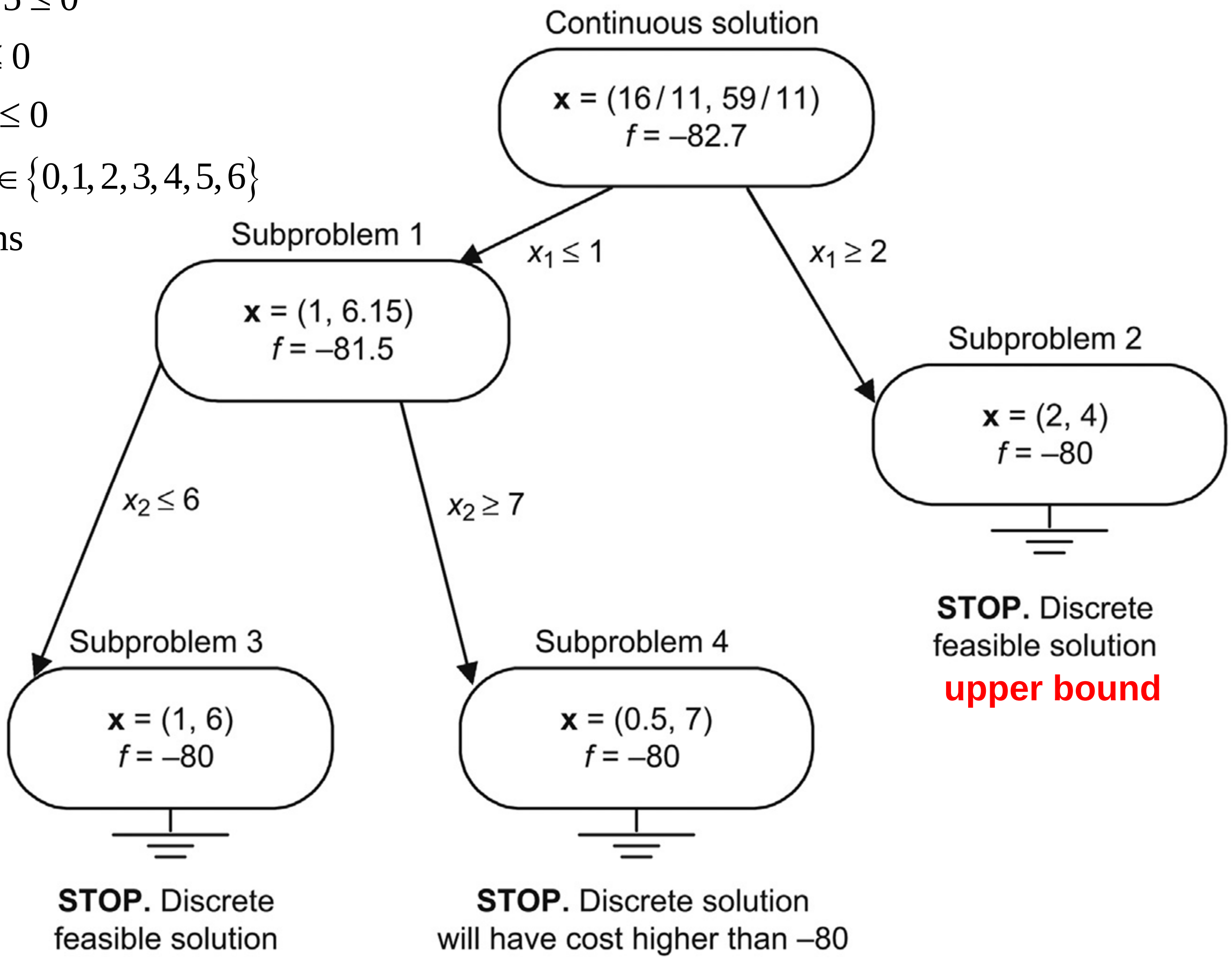
$$\text{subject to} \quad g_1 = -20x_1 - 10x_2 + 75 \le 0$$

$$g_2 = 12x_1 + 7x_2 - 55 \le 0$$

$$g_3 = 25x_1 + 10x_2 - 90 \le 0$$

$$x_1 \in \{0,1,2,3\} \text{ and } x_2 \in \{0,1,2,3,4,5,6\}$$

full enumeration: 28 combinations



BBM for General MV-OPT

- practical applications for nonlinear discrete problems
  - Functions: differentiable, design variables: nondiscrete
  - large number of discrete design variables → number of subproblems (nodes) becomes large
- strategies to reduce the number of nodes
  - variable that is used for branching is fixed to the assigned value: reduce dimensionality of the subproblem
  - early establishment of a good upper bound on the cost is important: choose an appropriate variable for branching
    - distance of a continuous variable from its nearest discrete value
    - cost function value when a variable is assigned a discrete value

Generalized Knapsack Problem

– The value carrying certain grocery items in a sack is to be maximized, subject to the constraint on the total weight.

$$\begin{aligned}
&\textit{Maximize} \quad 14x_1 + 12x_2 + 20x_3 + 8x_4 + 11x_5 + 7x_6 \\
&\text{subject to} \quad 1.1x_1 + 0.95x_2 + 1.2x_3 + 2x_4 + x_5 + x_6 \le 25 \\
&\qquad\qquad\quad x_i \ge 0,\ \ x_1, x_2, x_3, x_4 \text{ (integers)}
\end{aligned}$$

$$\left(x_1^* = 1, x_2^* = 0, x_3^* = 20, x_4^* = 0, x_5^* = 0.05, x_6^* = 0.0,\ f^* = 412.55\right)$$