Contents

- Basic concepts of solution methods
- Overview of deterministic methods
  - Covering method, Zooming method, Generalized Descent method / Tunneling method
- Overview of stochastic methods
  - multistart, clustering, control random search, acceptance–rejection (A–R), stochastic integration, stochastic zooming, and domain elimination
- Local-Global stochastic methods
- Numerical performance of methods

Basic Concepts (1)

- Definition of global minimum

$$\text{A point } \mathbf{x}^* \text{ is called a} \begin{cases} \textit{local} \text{ minimum if } f\left(\mathbf{x}^*\right) \leq f\left(\mathbf{x}\right) \text{ for all } \mathbf{x} \\ \text{in a small feasible neighborhood of the point } \mathbf{x}^* \\ \textit{global} \text{ minimum if } f\left(\mathbf{x}^*\right) \leq f\left(\mathbf{x}\right) \text{ for all } \mathbf{x} \text{ in the feasible set } S \end{cases}$$

$$S = \left\{x \mid h_i\left(\mathbf{x}\right) = 0,\ i = 1,\ldots,p;\ g_j\left(\mathbf{x}\right) \leq 0,\ j = 1,\ldots,m\right\}$$

$$S_b = \left\{x_i \mid x_{iL} \leq x_i \leq x_{iU},\ i = 1,\ldots,n\right\}$$

- Assumptions
  - Continuous variables and functions
  - Unconstrained problem only
    - Constraints can be treated implicitly using the penalty or augmented Lagrangian methods
    - Can terminate at infeasible points

Basic Concepts (2)

- Treat or require explicit bound constraints on design variables
- Repeatedly search for local minima in their algorithm
  - robust and efficient software to search for local minima
- Characterization of a global minimum
  - Existence: Weierstrass Theorem guarantees
  - Finding: different matter
  - Local optimum point: KKT necessary conditions
  - no mathematical conditions that characterize a global minimum point, except when the problem can be shown to be convex
  - How do we know that a numerical search process has terminated at a global minimum point? do NOT know
    - Difficult to define a precise stopping criterion

Overview of Methods

- Deterministic methods
  - Exhaustive search over the set S_b
  - Success can be guaranteed for only the function that satisfy certain conditions
  - covering, zooming, generalized descent, and tunneling
- Stochastic methods
  - Variations of pure random search
  - a random point is picked and manipulated or used before the next one is chosen
  - multistart, clustering, control random search, acceptance–rejection (A–R), stochastic integration, stochastic zooming, and domain elimination
  - nature-inspired methods

Deterministic Methods

- Find the global minimum by an exhaustive search over the set S_b
- Absolute guarantee of success?
  - Additional assumptions about the cost function: Lipschitz continuity condition for the function
  - There exists a Lipschitz constant L such that for all **x**, **y** in the set S_b, $\left| f(\mathbf{x}) - f(\mathbf{y}) \right| \le L \left\| \mathbf{x} - \mathbf{y} \right\|$
  - Upper bound on the rate of change of f(x) [Lipschitz constant] can be used in various ways to perform an exhaustive search
  - In practice, difficult to verify whether a function satisfies such a condition for all points in the set S_b
- Finite exact method vs. heuristic method
  - Finite exact method: absolute guarantee that the global minimum will be found in a finite (large) number of steps
  - Heuristic method: only an empirical guarantee

$$A_\varepsilon = \left\{ \mathbf{x} \in S \mid \left[f(\mathbf{x}) - \varepsilon \right] \le f(\mathbf{x}_G^*) \right\}$$

Covering Methods

- "cover" the set S_b by evaluating the cost function at all of the points in search for the global minimum
  - to define a mesh of points that may or may not be uniform for evaluating functions
  - mesh density is determined using the Lipschitz constant L
  - mesh points: centers of (hypercubes inscribed in the) hyperspheres
  - upper and lower bounds on the cost function over a subset of S_b are computed by interval arithmetic
  - successively form closer approximations (of given functions) that can be separated into convex and concave terms
  - Initial L(smaller approximation) and ε(larger value) → L increase and ε decrease → entire covering procedure is repeated until the difference between two consecutive solutions is less than ε
- generally not practical for problems having more than two variables

Zooming Method

- designed especially for problems with general constraints
- Reduce the specified target value to “zoom in” on the global minimum

- Basic idea
  - initiate the search for a constrained local minimum from any point
  - problem is redefined that the current solution is eliminated by adding the following constraint:

$$f(\mathbf{x}) \le \gamma f(\mathbf{x}^*) \quad \text{where} \begin{cases} 0 < \gamma < 1 \text{ if } f(\mathbf{x}^*) > 0 \\ \gamma > 1 \text{ if } f(\mathbf{x}^*) < 0 \end{cases}$$

  - process continued until no more minimum points can be found
- good alternative to stochastic and other methods
- Limitations
  - As the target level for global minimum of the cost function is lowered, a feasible set for the problem keeps on shrinking → disjointed feasible set

Methods of Generalized Descent (1)

- heuristic deterministic
- Decent steps along search directions (straight line)
  - More effective if we follow a curvilinear path in design space
- trajectory methods that generate curvilinear paths in the design space in search of minimum points
  - Trajectory: design history of the cost function from the starting point x(0) to a local minimum point x*

Let the design vector $\mathbf{x}$ be dependent on the parameter t that

monotonically increases along the solution curve $\mathbf{x}(t)$ and is zero at $\mathbf{x}^{(0)}$.

The simplest path from an arbitrary initial point $\mathbf{x}^{(0)}$ to $\mathbf{x}^*$

$\rightarrow$ continuous steepest-descent trajectory

$$\left.\begin{aligned} \dot{\mathbf{x}}(t) &= -\nabla f(\mathbf{x}) \\ \dot{\mathbf{x}}(t) &= -\left[\mathbf{H}(\mathbf{x})\right]^{-1} \nabla f(\mathbf{x}) \end{aligned}\right\} \text{ with } \mathbf{x}(t) = \mathbf{x}^{(0)}$$

Methods of Generalized Descent (2)

- generalized descent methods for global optimization
  - extensions of the trajectory methods: solutions for certain second-order differential equations
  - differential equations use the function values and function gradients along the trajectories
  - solution properties: trajectories pass through the majority of the stationary points of the cost function
- two types of generalized descent methods
  - Trajectory methods: modify the differential equation describing the local descent trajectory
  - Penalty methods: apply the standard local algorithm repeatedly to a modified cost function
    - algebraic functions, filled function, and tunneling

Trajectory Methods

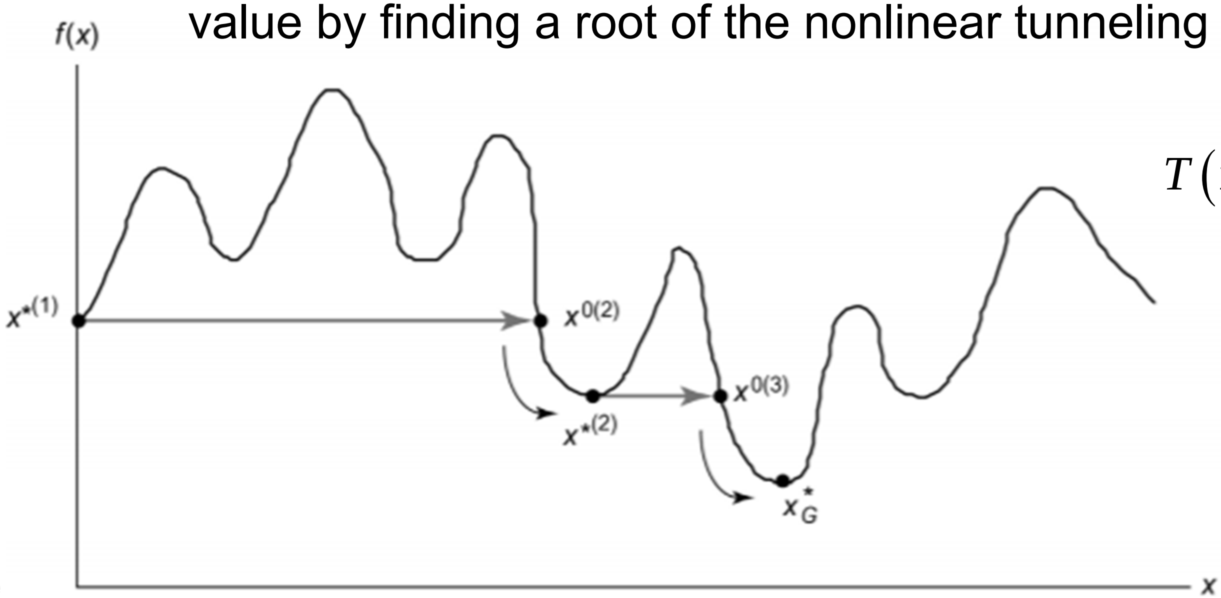
- Alternation of descents and ascents
  - descend to a local minimum from a starting point: combination of steepest–descent and Newton's methods
  - get from the local minimum point to a saddle point: eigenvector corresponding to the maximum eigenvalue of the Hessian as the search direction
  - pass through a saddle point: Newton's method
- Golf methods
  - analogous to the mechanics of inertial motion of a particle of mass m moving in a force field

$$m(t)\ddot{\mathbf{x}}(t)\underbrace{-n(t)\dot{\mathbf{x}}(t)}_{\substack{\text{dissipating or}\\ \text{nonconservative}\\ \text{resistance force}}} = -\nabla f(\mathbf{x}) \quad \text{where} \quad m(t) \ge 0 \text{ and } n(t) \le 0$$

$$\xrightarrow[n(t)=0]{m(t)=1} \ddot{\mathbf{x}}(t) = -\nabla f(\mathbf{x})$$

Tunneling Method (1)

- heuristic generalized descent penalty methods
- Basic idea
  - execute the two phases (local minimization and tunneling) successively until some stopping criterion is satisfied
  - tunneling phase
    - determine a starting point that is different from x* but has a cost function value smaller than or equal to the known minimum value by finding a root of the nonlinear tunneling function, T(x)



$$T(\mathbf{x}) = \frac{f(\mathbf{x}) - f(\mathbf{x}^*)}{\left[\sum_{i=1}^{n} \left(\mathbf{x}_i - \mathbf{x}_i^* \right)^2 \right]^p}$$

Tunneling Method (2)

- Characteristics
  - As the tunneling level approaches a global minimum, the number of computations increases because there are fewer roots of the tunneling function
    - This difficulty is similar to the one noted for the zooming method
  - Global descent property
    - local minima obtained by the minimization phase approaches the global minimum in an orderly fashion
    - efficient relative to other methods, especially for problems with a large number of local minima
    - advantage that a point with a smaller cost function value is reached at each and every iteration