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# Basic Concepts

- Problem Definition

$$\left.\begin{array}{l}\text{Minimizes } f(\mathbf{x}) = \left(f_1(\mathbf{x}), \ldots, f_k(\mathbf{x})\right) \\ \text{subject to } \begin{cases} h_i(\mathbf{x}) = 0; & i = 1, \ldots, p \\ g_j(\mathbf{x}) \le 0; & j = 1, \ldots, m \end{cases}\end{array}\right\} \text{ where } \begin{cases} k: \text{ number of objective functions} \\ p: \text{ number of equality constraints} \\ m: \text{ number of inequality constraints} \end{cases}$$

$f(x)$: $k$-dimensional vector of objective functions

$$S = \left\{x \mid h_i(\mathbf{x}) = 0,\ i = 1, \ldots, p;\ g_j(\mathbf{x}) \le 0,\ j = 1, \ldots, m\right\}$$

- Criterion Space and Design Space

| | |
|---|---|
| Design Space $\Leftrightarrow$ | Criterion (Cost) Space |
| $g_j(\mathbf{x}) = 0$ | $q_j$ |
| $S$ | attainable set $Z = \left\{f(\mathbf{x}) \mid \mathbf{x} \text{ in the feasible set } S\right\}$ |

Attainability: a point in the criterion space can be related to a point in the feasible design space

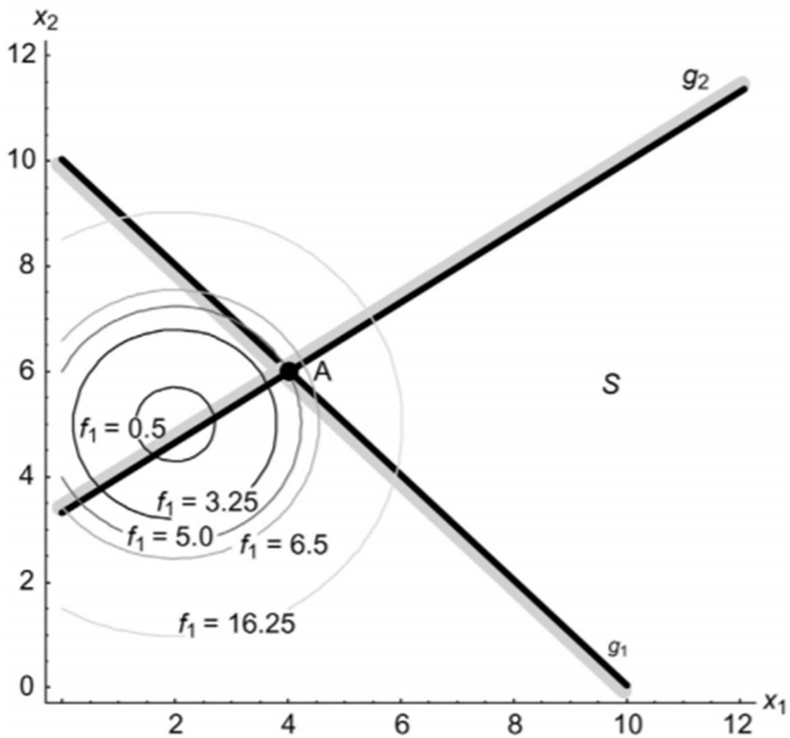
$$\left(\text{each point in the feasible design space}\right) \underset{\text{X}}{\overset{\text{O}}{\rightleftarrows}} \left(\text{a point in the criterion space}\right)$$

# Single vs. Two-Objective Optimization (1)

Minimize $f_1(\mathbf{x}) = (x_1 - 2)^2 + (x_2 - 5)^2$

subject to $\begin{cases} g_1 = -x_1 - x_2 + 10 \le 0 \\ g_2 = -2x_1 - 3x_2 - 10 \le 0 \end{cases}$

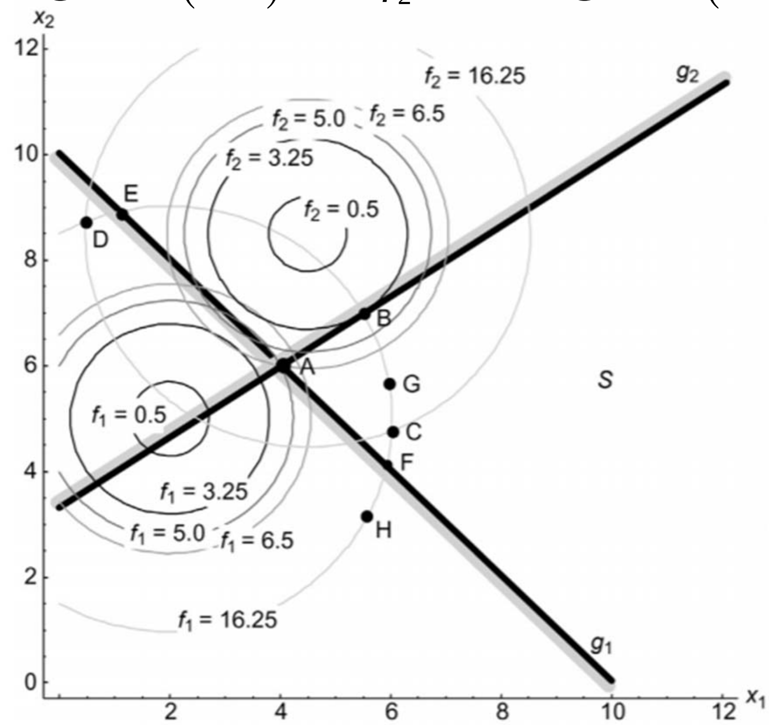
$\rightarrow f_1^* = 5 @ \mathbf{x}^* = (4, 6)$




Minimize $\begin{cases} f_1(\mathbf{x}) = (x_1 - 2)^2 + (x_2 - 5)^2 \\ f_2(\mathbf{x}) = (x_1 - 4.5)^2 + (x_2 - 8.5)^2 \end{cases}$

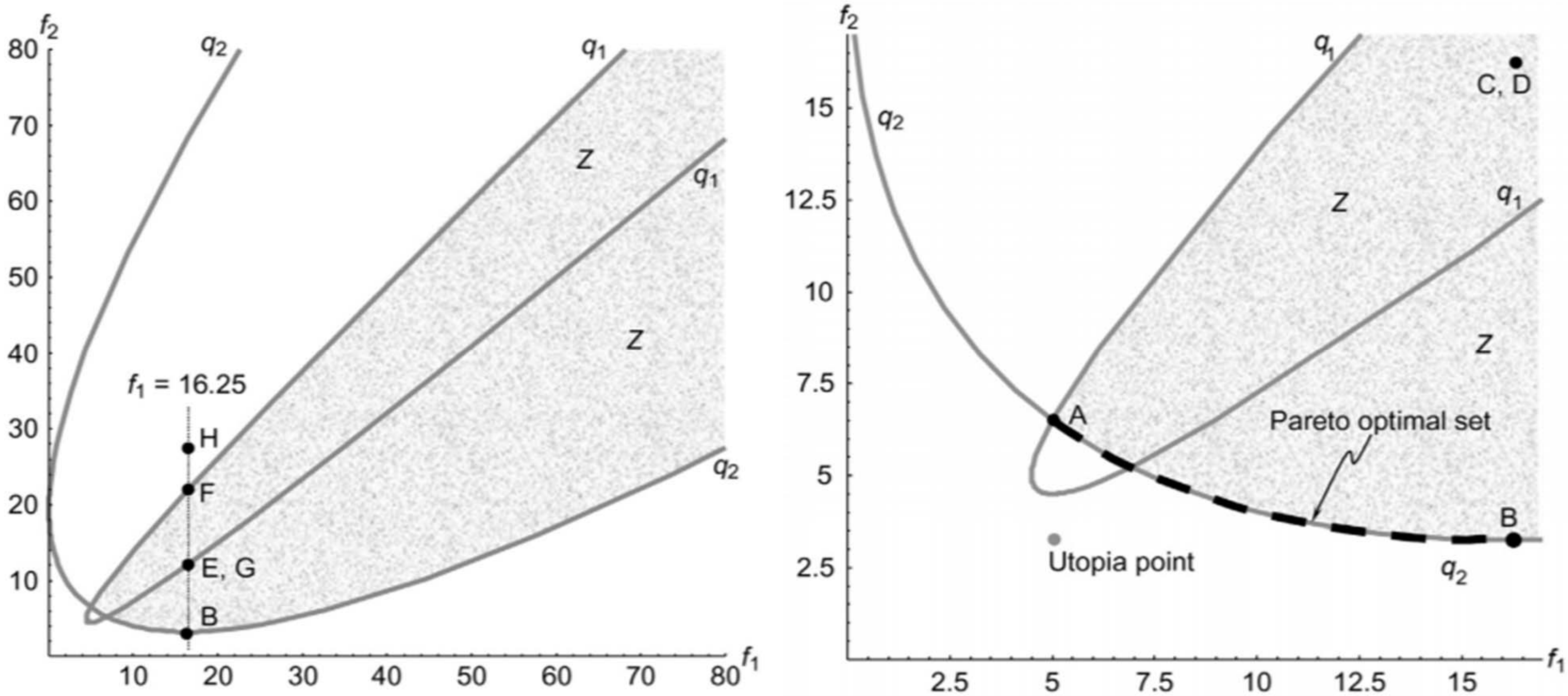
subject to $\begin{cases} g_1 = -x_1 - x_2 + 10 \le 0 \\ g_2 = -2x_1 - 3x_2 - 10 \le 0 \end{cases}$

$\rightarrow f_1^* = 5 @ \mathbf{x}^* = (4, 6)$ and $f_2^* = 3.25 @ \mathbf{x}^* = (5.5, 7.0)$



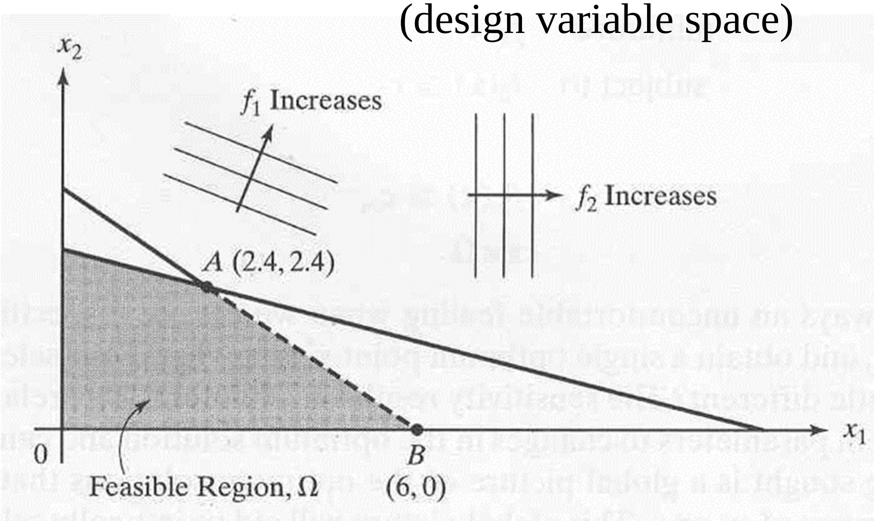
# Single vs. Two-Objective Optimization (2)

- for one objective function value, there may be many different feasible design points in the design space S
- how can point E be in the feasible criterion space?

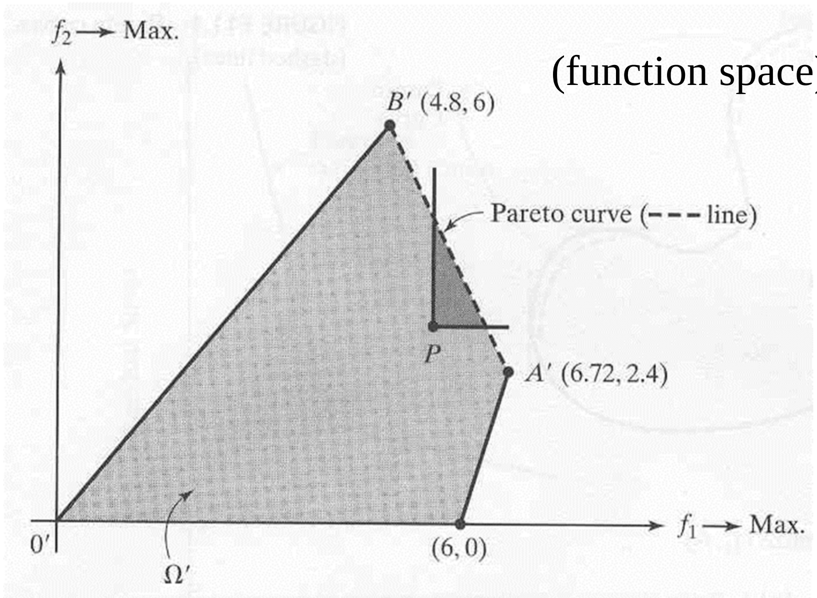


# Example

$$\left.\begin{array}{ll} \textit{Maximize} & f_1 = 0.8x_1 + 2x_2 \text{ and } f_2 = x_1 \\ \text{subject to} & 2x_1 + 3x_2 \le 12 \\ & x_1 + 4x_2 \le 12 \\ & x_i \ge 0 \end{array}\right\} \rightarrow \begin{cases} f_1^{\max} = 6.72 \,@\, x^* = (2.4, 2.4) \\ f_2^{\max} = 6.00 \,@\, x^* = (6.0, 0.0) \end{cases}$$

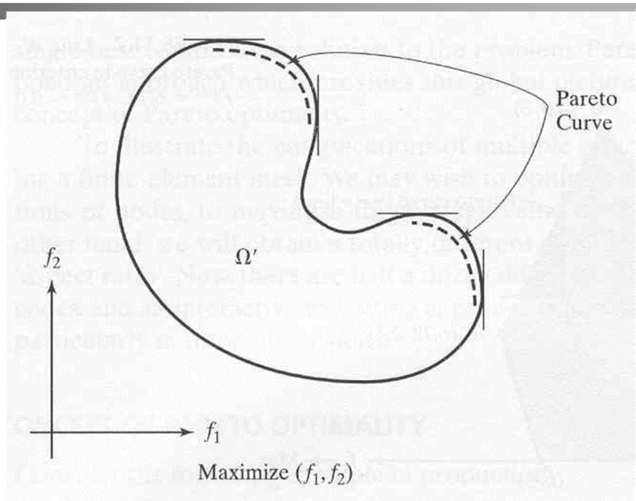


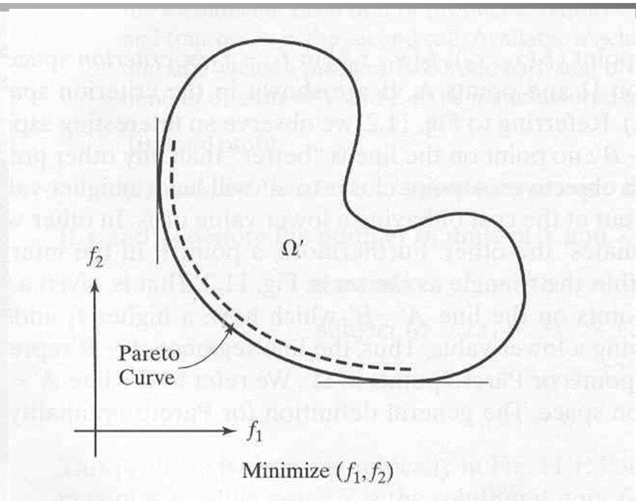
(design variable space)

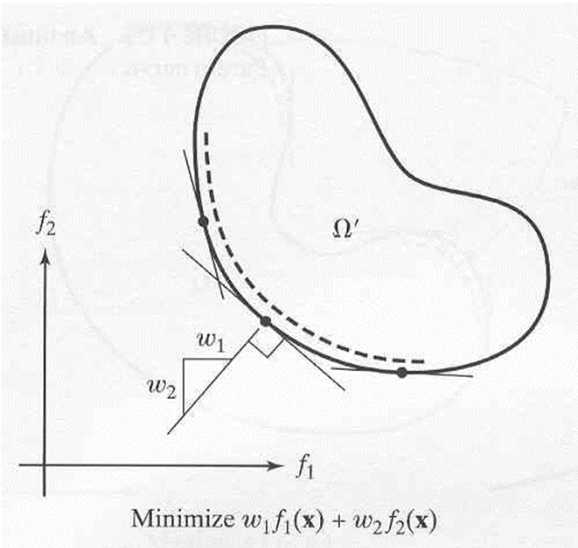


(function space)

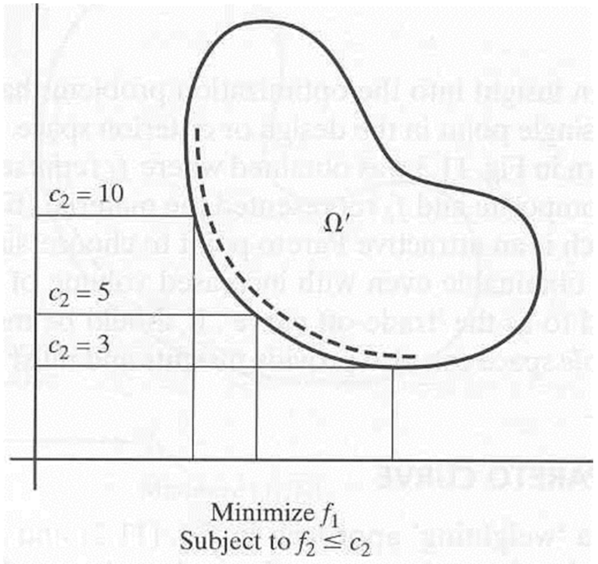
# Pareto Curves




(weighting)




(constraint)

# Solution Concepts (1)

- it is not clear what is meant by the minimum of multiple functions that may have opposing characteristics since what decreases the value of one function may increase the value of another

- Pareto optimality (Pareto, 1906)
  - A point x* in the feasible design space S is Pareto optimal if and only if there does not exist another point x in the set S such that $f(x) \leq f(x^*)$ with at least one $f_i(x) < f_i(x^*)$ [reduces at least one objective function without increasing another one]
  - Pareto optimal set is always on the boundary of the feasible criterion space Z, it is not necessarily defined by the constraints
    - Z exists even for unconstrained problems: Pareto optimal set is defined by the relationship between the gradients of the objective functions
- weakly Pareto optimal
  - "~" such that $f(x) < f(x^*)$: there is no point that improves all of the objective functions simultaneously; however, there may be points that improve some of the objectives while keeping others unchanged

# Solution Concepts (2)

- Efficiency
  - A point x* in the feasible design space S is efficient if and only if there does not exist another point x in the set S such that $f(x) \leq f(x^*)$ with at least one $f_i(x) < f_i(x^*)$. Otherwise, x* is inefficient. The set of all efficient points is called the efficient frontier.
- Dominance
  - A vector of objective functions $f^*=f(x^*)$ in the feasible criterion space Z is nondominated if and only if there does not exist another vector f in the set Z such that $f \leq f^*$, with at least one $f_i < f_i^*$. Otherwise, f* is dominated.
- Efficiency: points in the design space
- Nondominance: points in the criterion space
- Pareto optimality: both the design and the criterion spaces

# Solution Concepts (3)

- Utopia (Ideal) point
  - A point f° in the criterion space is called the utopia point if $f_i^\circ = \min\{f_i(x) \mid \text{for all } x \text{ in the set } S\}$, i = 1 to k
  - obtained by minimizing each objective function without regard for other objective function
  - general, it is not attainable
- Compromise solution
  - solution that is as close as possible to the utopia point
  - Pareto optimal

$$D(\mathbf{x}) = \left\| f(\mathbf{x}) - f^\circ \right\| = \left\{ \sum_{i=1}^{k} \left[ f_i(\mathbf{x}) - f_i^\circ \right]^2 \right\}^{1/2}$$

# Preferences and Utility Functions

- Preferences
  - infinitely many Pareto optimal solutions → make decisions concerning which solution is preferred
  - three approaches to expressing preferences about different objective functions
    - declared before solving the multi-objective optimization problem
    - indicated by interacting with the optimization routine and making choices based on intermediate optimization results
    - calculate the complete Pareto optimal set (or its approximation) and then select a single solution point after the problem has been solved
- Utility function
  - mathematical expression that attempts to model the decision maker's preferences

- Generation of Pareto Optimal Set
  - Some methods always yield Pareto optimal solutions but may skip certain points in the Pareto optimal set → obtain just one solution point
  - Other methods are able to capture all of the points in the Pareto optimal set, but may also provide non-Pareto optimal points → complete Pareto optimal set
- Normalization of Objective Functions

$$f_i^{\max}(\mathbf{x}) = \begin{cases} \max\limits_{1 \le j \le k} f_i\left(\mathbf{x}_j^*\right) \text{ where } \mathbf{x}_j^* : \text{point that minimizes the } j\text{-th objective function} \\ \text{absolute maximum value of } f_i(\mathbf{x}) \\ \text{or its approximation based on engineering intuition} \end{cases}$$

# Multi-objective Genetic Algorithms

  - ability to converge to the Pareto optimal set rather than a single Pareto optimal point
  - Niche: group of points that are close together (typically in terms of distance in the criterion space)
- How to
  - evaluate fitness
  - incorporate the idea of Pareto optimality
  - how to determine which potential solution points should be selected (will survive) for the next iteration (generation)
- Selection Strategy
  - Vector-Evaluated GA, Ranking, Pareto Fitness Function, Pareto-Set Filter, Elitist Strategy, Tournament Selection, Niche Techniques

# Selection Strategy (1)

- Vector-Evaluated GA
  - minimum of a single-objective function → vertices of the Pareto optimal set
  - for a problem with k objectives, k subsets are created, each with $N_p/k$ members
  - does not yield an even distribution of Pareto optimal points
- Ranking
  - give each design a rank based on whether it is dominated in the criterion space
  - Fitness is then based on a design's rank within a population
    - All nondominated points receive a rank of 1
    - points with a rank of one are temporarily removed from consideration, and the points that are nondominated relative to the remaining group are given a rank of 2
    - fitness is determined such that it is inversely proportional to rank

# Selection Strategy (2)

- Pareto Fitness Function

$$F(\mathbf{x}_i) = \max_{j \neq i;\, j \in P} \left[ \min_{1 \le s \le k} \left\{ f(\mathbf{x}_i) - f(\mathbf{x}_j) \right\} \right] : \text{ maximin fitness function}$$

$P$ : set of nondominated points in the current population

  - first determine all of the nondominated points before evaluating the fitness of the designs
  - nondominated points have negative fitness values
  - relatively simple and effective
- Pareto-Set Filter
  - stores two sets of solutions: the current population and the filter (another set of potential solutions, approximate Pareto set)
  - points with a rank of 1 are saved in the filter → nondominated check
  - When the filter is full, points at a minimum distance from other points are discarded

# Selection Strategy (3)

- Elitist Strategy
  - independently of the ranking scheme
  - user-specified number of points from the tentative set of nondominated solutions are reintroduced into the current population
  - k solutions with the best values for each objective function
- Tournament Selection
  - Two points(candidate points) are compared with each member of tournament (or comparison) set
  - size of the tournament set is prespecified as a percentage of the total population
  - imposes the degree of difficulty in surviving (domination pressure)

# Selection Strategy (4)

- Niche Techniques
  - methods for ensuring that a set of designs does not converge to a niche
  - foster an even spread of points (in the criterion space).
  - genetic (or population) drift: a limited number of niches → converge to or cluster around a limited set of Pareto optimal points
  - Fitness sharing: penalize the fitness of points in crowded areas, thus reducing the probability of their survival for the next iteration

# Weighted Sum Method

$$U = \sum_{i=1}^{k} w_i f_i(\mathbf{x}) \text{ where } \sum_{i=1}^{k} w_i = 1 \text{ and } \mathbf{w} > 0$$

- Weight determination
  - set w to reflect preferences before the problem is solved
  - systematically alter w to yield different Pareto optimal points (to generate the Pareto optimal set)
- Difficulties
  - satisfactory a priori weight selection does not necessarily guarantee that the final solution will be acceptable; one may have to re-solve the problem with different weights
  - impossible to obtain points on nonconvex portions of the Pareto optimal set in the criterion space
  - varying the weights consistently and continuously may not necessarily result in an even distribution of Pareto optimal points and an accurate, complete representation of the Pareto optimal set

# Weighted Min-Max Method

$$U = \max_i \left\{ w_i \left[ f_i(\mathbf{x}) - f_i^{\circ} \right] \right\} \rightarrow \begin{cases} \text{minimize } \lambda \\ w_i \left[ f_i(\mathbf{x}) - f_i^{\circ} \right] - \lambda \leq 0; \; i = 1\ldots, k \end{cases}$$

- Advantages
  - Provide a clear interpretation of minimizing the largest difference between $f_i(x)$ and $f_i^{\circ}$
  - Provide all of the Pareto optimal points
  - always provides a weakly Pareto optimal solution
  - relatively well suited for generating the complete Pareto optimal set (with variation in the weights)
- Disadvantages
  - requires the minimization of each objective to obtain the utopia point, which can be computationally expensive
  - requires that additional constraints be included
  - not clear exactly how to set the weights when only one solution point is desired

# Weighted Global Criterion Method (1)

$$U = \left\{ \sum_{i=1}^{k} \left( w_i \left[ f_i(\mathbf{x}) - f_i^{\circ} \right] \right)^p \right\}^{1/p} \quad \text{where } \sum_{i=1}^{k} w_i = 1 \text{ and } \mathbf{w} > 0$$

$$p = \begin{cases} 1: \text{weighted sum with the objective functions adjusted with the utopia point} \\ 2(\mathbf{w} = 1): \text{distance from the utopia point} \\ \infty \rightarrow U = \max_i \left\{ w_i \left[ f_i(\mathbf{x}) - f_i^{\circ} \right] \right\} \end{cases}$$

- increasing the value of p can increase its effectiveness in providing the complete Pareto optimal set → This explains why the weighted min–max approach can provide the complete Pareto optimal set with variation in the weights
- utopia point → z not in Z (aspiration point, reference point, goal, or target point), U (achievement function)
- utopia point, or compromise-programming methods

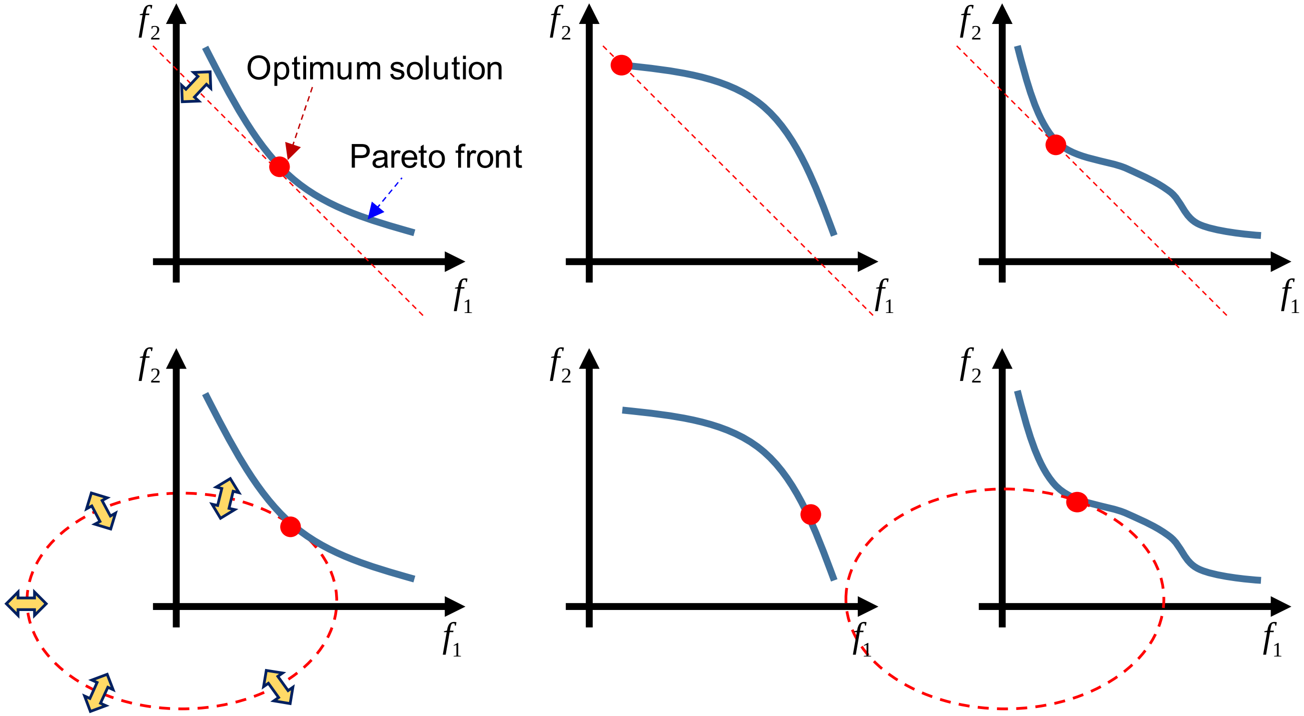
# Weighted Global Criterion Method (2)

Case 1: $p = 1$

$w_1 f_1 + w_2 f_2$

Case 2: $p = 2$

$\left\{ (w_1 f_1)^2 + (w_2 f_2)^2 \right\}^{1/2}$



# Weighted Global Criterion Method (3)

- Advantages
  - gives a clear interpretation of minimizing the distance from the utopia point (or the aspiration point)
  - gives a general formulation that reduces to many other approaches
  - allows multiple parameters to be set to reflect preferences
  - always provides a Pareto optimal solution when the utopia point is used
- Disadvantages
  - use of the utopia point requires minimization of each objective function, which can be computationally expensive
  - use of an aspiration point requires that it be infeasible in the criterion space in order to yield a Pareto optimal solution
  - setting of parameters is not intuitively clear when only one solution point is desired

# Lexicographic Method

$$\text{for } i = 1,\ldots,k$$

$$\begin{cases} \text{Minimize } f_i(\mathbf{x}) \\ \text{subject to } f_j(\mathbf{x}) \le f_j(\mathbf{x}_j^*);\ \ j = 1,\ldots,(i-1),\ i > 1,\ i = 1,\ldots,k \end{cases}$$

– preferences are imposed by ordering the objective functions according to their importance or significance, rather than by assigning weights

- Advantages
  – unique approach to specifying preferences
  – does not require that the objective functions be normalized
  – always provides a Pareto optimal solution
- Disadvantages
  – can require the solution of many single-objective problems to obtain just one solution point
  – requires that additional constraints be imposed
  – most effective when used with a global optimization engine

# Bounded Objective Function Method

$$\begin{cases} \text{Minimize } f_s(\mathbf{x}) \\ \text{subject to } l_i \le f_i(\mathbf{x}) \le \varepsilon_i; \quad i = 1, \ldots, k, \; i \ne s \end{cases}$$

$f_s(\mathbf{x})$ : single most important objective function

$l_i \le f_i(\mathbf{x}) \le \varepsilon_i \rightarrow f_i(\mathbf{x}) \le \varepsilon_i$ [$\varepsilon$-constaint appoach]

guideline for selecting $\varepsilon_i$ : $f_i(\mathbf{x}_i^*) \le \varepsilon_i \le f_s(\mathbf{x}_i^*)$

- Advantages
  - focuses on a single objective with limits on others
  - always provides a weakly Pareto optimal point, assuming that the formulation gives a solution
  - not necessary to normalize the objective functions
  - gives Pareto optimal solution if one exists and is unique
- Disadvantages
  - optimization problem may be infeasible if the bounds on the objective functions are not appropriate

# Goal Programming

$$\begin{cases} \text{Minimize} & \sum_{i=1}^{k} \left( d_i^+ + d_i^- \right) \\ \text{subject to} & \begin{cases} f_j(\mathbf{x}) + d_j^+ + d_j^- = b_i \\ d_j^+,\ d_j^- \geq 0,\ d_j^+ d_j^- = 0 \end{cases} ;\quad i = 1, \ldots, k \end{cases}$$

$d_j$ : deviation from the goal $b_j$ for the $j$-th objective function

$b_j = f_j^\circ \rightarrow$ global criterion method

- Advantages
  - easy to assess whether the predetermined goals have been reached
  - easy to tailor the method to a variety of problems not necessary to normalize the objective functions
- Disadvantages
  - no guarantee that the solution is even weakly Pareto optimal
  - increase in the number of variables
  - increase in the number of constraints

# Selection of Methods

| Method | Always yields Pareto optimal point? | Can yield all Pareto optimal points? | Involves weights? | Depends on function continuity? | Uses utopia point? |
|---|---|---|---|---|---|
| Genetic | Yes | Yes | No | No | No |
| Weighted sum | Yes | No | Yes | Problem type and optimization engine determines this | Utopia point or its approximation is needed for function normalization or in the formulation of the method |
| Weighted min–max | Yes[a] | Yes | Yes | Same as above | Same as above |
| Weighted global criterion | Yes | No | Yes | Same as above | Same as above |
| Lexicographic | Yes[b] | No | No | Same as above | No |
| Bounded objective function | Yes[c] | No | No | Same as above | No |
| Goal programming | No | No | No[d] | Same as above | No |

[a] *Sometimes solution is only weakly Pareto optimal.*
[b] *Lexicographic method always provides Pareto optimal solution only if global optimization engine is used or if solution point is unique.*
[c] *Always weak Pareto optimal if it exists; Pareto optimal if solution is unique.*
[d] *Weights may be incorporated into objective function to represent relative significance of deviation from particular goal.*