Duality in Nonlinear Programming

- Given a nonlinear programming problem, there is another nonlinear programming problem closely associated with it
  - primal problem / dual problem
  - the same optimum objective function values under certain convexity assumptions: local convexity, *local duality theory*
  - solve the primal problem indirectly by solving the dual problem

Local Duality: Equality Constraints

[Problem E]

$$\underset{\mathbf{x}}{\text{Minimize}} \quad f(\mathbf{x})$$

subject to

$$h_i(\mathbf{x}) = 0,\ i = 1, \ldots, p$$

[Dual Function]

near $\mathbf{v}^*$, minimum is taken locally with respect to $\mathbf{x}$ near $\mathbf{x}^*$

$$\phi(\mathbf{v}) = \underset{\mathbf{x}}{\text{Minimize}}\ L(\mathbf{x}, \mathbf{v}) \left[= f(\mathbf{x}) + \sum_{i=1}^{p} v_i h_i \right]$$

$\longleftrightarrow$

[Dual Problem]

$$\underset{\mathbf{v}}{\text{Maximize}} \quad \phi(\mathbf{v})$$

constrained problem in $\mathbf{x} \leftrightarrow$ unconstrained problem in $\mathbf{v}$

$$L(\mathbf{x}, \mathbf{v}) = f(\mathbf{x}) + \sum_{i=1}^{p} v_i h_i = f(\mathbf{x}) + (\mathbf{v} \cdot \mathbf{h})$$

$$\mathbf{H}_x(\mathbf{x}, \mathbf{v}) = \frac{\partial^2 L}{\partial \mathbf{x}^2} = \frac{\partial^2 f(\mathbf{x})}{\partial \mathbf{x}^2} + \sum_{i=1}^{p} v_i \frac{\partial^2 h_i}{\partial \mathbf{x}^2};\ \mathbf{N} = \left[\frac{\partial h_j}{\partial x_i} \right]_{n \times p}$$

assumption: $\mathbf{H}_x(\mathbf{x}^*, \mathbf{v}^*)$ positive definite $\rightarrow L(\mathbf{x}^*, \mathbf{v}^*)$ is locally convex

$\rightarrow \mathbf{x}^*$ to be an isolated local minimum of Problem E

and a local minimum for the unconstrained problem $\left[\underset{\mathbf{x}}{\text{minimize}}\, L(\mathbf{x}, \mathbf{v}^*) \right]$

vicinity of $(\mathbf{x}^*, \mathbf{v}^*)$: $\dfrac{\partial L(\mathbf{x}, \mathbf{v})}{\partial \mathbf{x}} = 0 \rightarrow \underset{\mathbf{x}}{\text{minimize}}\, L(\mathbf{x}, \mathbf{v})$

Lemmas and Theorem

[Lemma]

The gradient of the dual function $\phi(\mathbf{v})$ is given as $\dfrac{\partial \phi(\mathbf{v})}{\partial \mathbf{v}} = \mathbf{h}(\mathbf{x}(\mathbf{v}))$

$\rightarrow$ simple to calculatae

The Hessian of the dual function $\phi(\mathbf{v})$ is given as $\mathbf{H}_v = \dfrac{\partial^2 \phi(\mathbf{v})}{\partial \mathbf{v}^2} = -\mathbf{N}^T \left[\mathbf{H}_x(\mathbf{x})\right]^{-1} \mathbf{N}$

$\rightarrow$ negative definite

[Local Duality Theorem]

For Problem E, let $\begin{cases} \mathbf{x}^* \text{ be a local minimum and a regular point} \\ \mathbf{v}^* \text{ be the Lagrange multipliers at } \mathbf{x}^* \\ \mathbf{H}_x(\mathbf{x}^*, \mathbf{v}^*) \text{ be positive definite} \end{cases}$

Then the dual problem $\left[\text{Maximize } \phi(\mathbf{v})\right]$ has a local solution at $\mathbf{v}^*$ with $\mathbf{x}^* = \mathbf{x}(\mathbf{v}^*)$.

The maximum value of the dual function is equal to the minimum value of $f(\mathbf{x})$;

that is, $\phi(\mathbf{v}^*) = f(\mathbf{x}^*)$

Example 5.7

- Consider the following problem in two variables; derive the dual of the problem and solve it:

$$\underset{x_1, x_2}{\text{Minimize}} \quad f(x_1, x_2) = -x_1 x_2$$

$$\text{subject to} \quad h(x_1, x_2) = (x_1 - 3)^2 + x_2^{\,2} = 5$$

$$\rightarrow x_1^* = 4, \; x_2^* = 2, \; v^* = 1, \; f^* = -8$$

Local Duality: Inequality Constraints

[Problem P]

$$\underset{\mathbf{x}}{\text{Minimize}} \quad f(\mathbf{x})$$

subject to

$$h_i(\mathbf{x}) = 0,\ i = 1, \ldots, p$$

$$g_j(\mathbf{x}) = 0,\ j = 1, \ldots, m$$

[Dual Function]

near $\mathbf{v}^*, \mathbf{u}^*$, minimum is taken locally with respect to $\mathbf{x}$ near $\mathbf{x}^*$

$$\phi(\mathbf{v},\mathbf{u}) = \underset{\mathbf{x}}{\text{Minimize}}\ L(\mathbf{x},\mathbf{v},\mathbf{u}) \left[= f(\mathbf{x}) + \sum_{i=1}^{p} v_i h_i + \sum_{j=1}^{m} u_j g_j \right],\ u_j \geq 0$$

$\longleftrightarrow$

[Dual Problem]

$$\underset{\mathbf{v},\mathbf{u}}{\text{Maximize}} \quad \phi(\mathbf{v},\mathbf{u})$$

$$u_j \geq 0,\ j = 1, \ldots, m$$

constrained problem in $\mathbf{x} \leftrightarrow$ unconstrained problem in $\mathbf{v}, \mathbf{u}$

$$L(\mathbf{x},\mathbf{v},\mathbf{u}) = f(\mathbf{x}) + \sum_{i=1}^{p} v_i h_i + \sum_{j=1}^{m} u_j g_j = f(\mathbf{x}) + (\mathbf{v} \cdot \mathbf{h}) + (\mathbf{u} \cdot \mathbf{g}),\ u_j \geq 0$$

Duality Theorem (1)

[Strong Duality Theorem]

For Problem P, let $\begin{cases} \mathbf{x}^* \text{ be a local minimum and a regular point} \\ \mathbf{v}^*, \mathbf{u}^* \text{ be the Lagrange multipliers at the optimum point } \mathbf{x}^* \\ \mathbf{H}_x\left(\mathbf{x}^*, \mathbf{v}^*, \mathbf{u}^*\right) \text{ be positive definite} \end{cases}$

Then $\mathbf{v}^*$ and $\mathbf{u}^*$ solve the dual problem $\left[\text{Maximize } \phi(\mathbf{v}, \mathbf{u})\right]$ with

$f\left(\mathbf{x}^*\right) = \phi\left(\mathbf{v}^*, \mathbf{u}^*\right)$ and $\mathbf{x}^* = \mathbf{x}\left(\mathbf{v}^*, \mathbf{u}^*\right)$

if $\mathbf{H}_x\left(\mathbf{x}^*, \mathbf{v}^*, \mathbf{u}^*\right)$ is NOT positive definite

$\rightarrow$ [Weak Duality Theorem]

Let $\mathbf{x}$ be a feasible solution for Problem P and let $\mathbf{v}$ and $\mathbf{u}$ be the feasible solution for the dual problem; thus, $h_i(\mathbf{x}) = 0,\ i = 1, \ldots, p$ and $g_j(\mathbf{x}) = 0,\ j = 1, \ldots, m$

Then $\phi(\mathbf{v}, \mathbf{u}) \le f(\mathbf{x})$

Duality Theorem (2)

1. Minimum $\left[f(\mathbf{x}) \text{ with } \mathbf{x} \in S \right] \geq$ Maximum $\left[\phi(\mathbf{v},\mathbf{u}) \text{ with } u_j \geq 0,\ j = 1,\ldots,m \right]$

2. If $f(\mathbf{x}^*) = \phi(\mathbf{v}^*,\mathbf{u}^*)$ with $u_j \geq 0,\ j = 1,\ldots,m$ and $\mathbf{x}^* \in S$,

then $\mathbf{x}^*$ and $(\mathbf{v}^*,\mathbf{u}^*)$ solve the primal and dual problem, respectively.

3. If Minimum $\left[f(\mathbf{x}) \text{ with } \mathbf{x} \in S \right] = -\infty,$ then the dial is infeasible,

and vice versa (ie, dual is infeasible, the primal is unbounded)

4. If Maximum $\left[\phi(\mathbf{v},\mathbf{u}) \text{ with } u_j \geq 0,\ j = 1,\ldots,m \right] = \infty,$

then the primal problem has no feasible solution,

and vice versa (ie, primal is infeasible, the dual is unbounded)

Saddle Point Theorem

$[\text{Lemma}]\begin{cases}\text{lower bound for primal cost function} \\ \text{upper bound for dual cost function}\end{cases}$

For any $\mathbf{v}^*$ and $\mathbf{u}^*$ with $u_j \geq 0,\ j = 1, \ldots, m$

$\phi(\mathbf{v}, \mathbf{u}) \leq f(\mathbf{x}^*)$

$[\text{Saddle Point Theorem}]$

For Problem P let all functions be twice continuously differentiable and

let $L(\mathbf{x}, \mathbf{v}, \mathbf{u})$ be defined as $L(\mathbf{x}, \mathbf{v}, \mathbf{u}) = f(\mathbf{x}) + (\mathbf{v} \cdot \mathbf{h}) + (\mathbf{u} \cdot \mathbf{g});\ u_j \geq 0,\ j = 1, \ldots, m$

Let $L(\mathbf{x}^*, \mathbf{v}^*, \mathbf{u}^*)$ exist with $u_j^* \geq 0,\ j = 1, \ldots, m.$ Also let $\mathbf{H}_x(\mathbf{x}^*, \mathbf{v}^*, \mathbf{u}^*)$ be positive definite.

Then $\mathbf{x}^*$ satisfying a suitable constraint qualification is a local minumum of Problem P

if and only if $(\mathbf{x}^*, \mathbf{v}^*, \mathbf{u}^*)$ is a saddle point of the Lagrangian; that is,

$L(\mathbf{x}^*, \mathbf{v}, \mathbf{u}) \leq L(\mathbf{x}^*, \mathbf{v}^*, \mathbf{u}^*) \leq L(\mathbf{x}, \mathbf{v}^*, \mathbf{u}^*)$

for all $\mathbf{x}$ near $\mathbf{x}^*$ and all $(\mathbf{v}, \mathbf{u})$ near $(\mathbf{v}^*, \mathbf{u}^*)$, with $u_j \geq 0,\ j = 1, \ldots, m$