Linear Programming (LP) Problem

- Constrained optimization
- “Liner”: the objective and the constraints
- “Programming”: scheduling or setting an agenda
- History
  - Optimal allocation of resources in the 1930s by economists
  - George B. Dantzig (1947): simplex method
    - Air Force Group during World War II
  - Revolutionary development to make optimal decisions in complex situations
    - Four Nobel Prizes related to LP
  - Karmarkar (1984)
    - Interior approach

Applications of LP

- Diet decisions, transportation, production and manufacturing, product mix, engineering limit analysis in design, airline scheduling, NLP(SLP)
  - Petroleum refineries
    - A mix of the purchased crude oil and the manufactured products that gives the maximum profit
  - Production plan in a manufacturing firm
    - Various cost and loss factors
  - Food processing industry
    - Optimal shipping plan for the distribution of a particular product from different manufacturing plants to various warehouses
  - Iron and steel industry
    - Decide the types of products to be made in their rolling mills
  - Routing
    - Optimal routing of messages in a communication network / aircraft and ships

Standard LP Definition (1)

- Minimization of a function with equality constraints and nonnegativity of design variables

$$\text{Minimize} \quad f = \sum_{i=1}^{n} c_i x_i$$

$$\text{subject to} \quad \sum_{j=1}^{n} a_{ij} x_j = b_i; \qquad i = 1, \ldots, m$$

$$x_j \geq 0; \qquad j = 1, \ldots, n$$

$$\left(b_i \geq 0 : \text{resource limits, } c_i \text{ and } a_{ij} : \text{known constants} \right)$$

$$\begin{aligned} &\text{Minimize} && f = \mathbf{c}^T \mathbf{x} \\ &\text{subject to} && \mathbf{A}\mathbf{x} = \mathbf{b} \\ & && \mathbf{x} \geq 0 \end{aligned}$$

Standard LP Definition (2)

- Linear constraints
  - Inequality: nonnegative slack variable s_i ($s_i \geq 0$)
    - Why not s_i^2 ? (nonlinear)
  - Treatment of "≤ type" / "≥ type" constraints

$$\begin{cases} 2x_1 - x_2 \leq 4 \rightarrow 2x_1 - x_2 + s_1 = 4 \quad (s_1 \geq 0) \\ -x_1 + 2x_2 \geq 2 \rightarrow -x_1 + 2x_2 - s_1 = 2 \quad (s_1 \geq 0) \end{cases}$$

- Unrestricted variables in sign
  - All design variables to be nonnegative

$$x_j = x_j^+ - x_j^- = \begin{cases} \text{nonnegative} : x_j^+ \geq x_j^- \\ \text{nonpositive} : x_j^+ \leq x_j^- \end{cases}$$

$$x_j^+ \geq 0 \text{ and } x_j^- \geq 0$$

Example 8.1

$$\text{Maximize} \quad z = \sum_{i=1}^{n} d_i x_i \Leftrightarrow \text{Minimize} \quad f = -\sum_{i=1}^{n} d_i x_i$$

$$\left.\begin{array}{ll} \text{Maximize} & z = 2y_1 + 5y_2 \\ \text{subject to} & 3y_1 + 2y_2 \le 12 \\ & -2y_1 - 3y_2 \le -6 \\ & y_1 \ge 0,\ y_2 \ : \text{unrestricted} \end{array}\right\} \rightarrow \left\{\begin{array}{ll} \underset{\mathbf{x}}{\text{Minimize}} & f = \begin{bmatrix} -2 & -5 & 5 & 0 & 0 \end{bmatrix} \mathbf{x}^T \\ \text{subject to} & \begin{bmatrix} 3 & 2 & -2 & 1 & 0 \\ 2 & 3 & -3 & 0 & -1 \end{bmatrix} \mathbf{x}^T \le \begin{bmatrix} 12 \\ 6 \end{bmatrix} \end{array}\right.$$

Basic Concepts

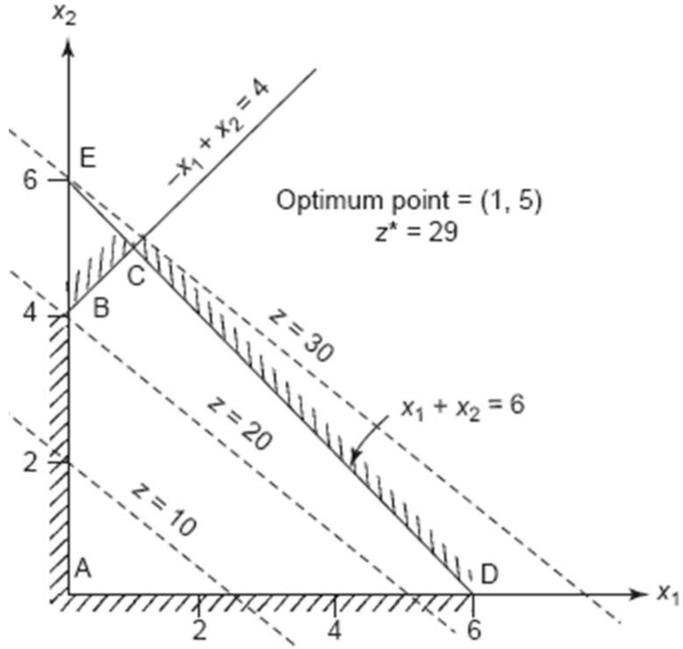
- LP problem is convex. If an optimum solution exists, it is global.
  - Feasible region (constraint set) is convex
  - Cost function is linear, so it is convex
- Solution always lies on the boundary of the feasible region if it exists.
  - For an unconstrained optimum, contradiction: $\frac{\partial f}{\partial x_i} = 0 \rightarrow c_i = 0$
- Optimum solution must satisfy equality constraints $\rightarrow$ more than one solution ($m < n$)
  - Infinite solutions $\rightarrow$ feasible solution that minimizes the cost function

LP Terminology

- Vertex (extreme) point
  - A point of the set that does not lie on a line segment
- Feasible solution
  - Any solution of the constraint equations satisfying the nonnegative conditions
- Basic (feasible) solution
  - By setting "redundant number" (n-m) of variables (nonbasic) to zero
- Degenerate basic (feasible) solution
  - If a basic (feasible) variable has zero value
- Optimum (basic) solution
  - Feasible solution minimizing the cost function
- Convex polyhedron: bounded feasible region
- Basis
  - Columns of coefficient matrix of constraint equations corresponding to basic variables (m-dimensional vector space)

Example 8.3

$$\left.\begin{aligned} \text{Maximize} \quad & z = 4x_1 + 5x_2 \\ \text{subject to} \quad & -x_1 + x_2 \le 4 \\ & x_1 + x_2 \le 6 \\ & x_1, x_2 \ge 0 \end{aligned}\right\} \rightarrow$$



No.	x_1	x_2	x_3	x_4	f	Location in Fig. 6-2
1	0	0	4	6	0	A
2	0	4	0	2	–20	B
3	0	6	–2	0	—	infeasible
4	–4	0	0	10	—	infeasible
5	6	0	10	0	–24	D
6	1	5	0	0	–29	C

Optimum Solution for LP Problems

- The collection of feasible solutions for an LP problem constitutes a convex set whose extreme points correspond to basic feasible solution
- Let rank(A) = m ($m \times n$ coefficient matrix A),
  - If there is a feasible solution, there is a basic feasible solution.
    - There must be at least one extreme point or vertex of convex feasible set
  - If there is an optimum feasible solution, there is an optimum basic feasible solution.
    - at least at one of the vertices of the *convex polyhedron* representing all of the feasible solutions
    - Optimum solution must be one of the basic feasible solutions
    - Search for optimum only among the basic feasible solutions ${}_nC_{(n-m)}$

Simplex Method (1)

- Proceed from one basic feasible solution to another in a way to continuously decrease the cost function until the minimum is reached
  - Gauss-Jordan elimination process
  - Simplex
    - Geometric figure formed by a set of (n+1) points in an n-dimensional space
    - A convex hull of any (n+1) points which do not lie on one hyperplane
    - the smallest convex set containing all the points, convex set
    - 2D: triangular, 3D: tetrahedron
  - Canonical form
    - Each equation has a variable (w/ unit coefficient) that does not appear in any other equation

Simplex Method (2)

$$\left.\begin{aligned}
x_1 + \cdots \quad + a_{1,m+1}x_{m+1} + \cdots + a_{1,n}x_n &= b_1 \\
x_2 + \cdots + a_{2,m+1}x_{m+1} + \cdots + a_{2,n}x_n &= b_2 \\
&\vdots \\
x_m + a_{m,m+1}x_{m+1} + \cdots + a_{m,n}x_n &= b_m
\end{aligned}\right\} \Rightarrow \begin{cases}
\boldsymbol{I}_{(m)}\boldsymbol{x}_{(m)} + \boldsymbol{Q}\boldsymbol{x}_{(n-m)} = \boldsymbol{b} \\
\boldsymbol{x}_{(n-m)} = 0 \rightarrow \boldsymbol{x}_{(m)} = \boldsymbol{b} \\
\text{(nonbasic)} \quad \text{(basic)}
\end{cases}$$

- Tableau: representation of a scene or a picture
  - Identify nonbasic / basic variables → basic solutions
- Pivot step
  - Starting from a basic feasible solution, find another one to reduce the cost
  - Interchanging a current basic variable w/ a nonbasic variable

Basic Steps

- (1) initial basic feasible solution (vertex)
  - Slack variables as basic and original variables as nonbasic
  - Cost function expressed in terms of only the <u>nonbasic variables</u>
- (2) check if it is the optimum point ?
  - all coefficients in the cost row become nonnegative
- (3) interchange a current basic variable w/ a nonbasic variable
  - Find a new basic feasible solution
  - Unbounded: all entries in the pivot column are negative
  - Select a nonbasic variable $\rightarrow$ pivot column ? (negative reduced cost coeff.)
  - Select a basic variable $\rightarrow$ pivot row ? (smallest ratio)
  - Complete the pivot step using the Gauss-Jordan elimination procedure
- (4) repeat until it satisfies (2)
  - Multiple optimum solutions: if a reduced cost coeff. corresponding to a nonbasic variable is zero in the final tableau

Selection of a nonbasic variable

- Main idea $\rightarrow$ to improve the design
  - reduce the current value of the cost function

$$x_i = b_i - \sum_{j=m+1}^{n} a_{ij} x_j; \quad i = 1, \ldots, m$$

$$f = \sum_{i=1}^{n} c_i x_i = \sum_{i=1}^{m} c_i \left(b_i - \sum_{j=m+1}^{n} a_{ij} x_j \right) + \sum_{j=m+1}^{n} c_j x_j = \underbrace{\sum_{i=1}^{m} c_i b_i}_{f_0} + \sum_{j=m+1}^{n} \underbrace{\left(c_j - \sum_{i=1}^{m} a_{ij} c_i \right)}_{c'_j} \underbrace{x_j}_{\text{nonbasic variables}}$$

$c'_j < 0 \rightarrow f \downarrow$

more than one negative $c'_j \rightarrow$ choose the nonbasic variable with the smallest c'_j

$$\left.\begin{array}{l} c'_j = 0 \rightarrow \text{no change in } f \\ c'_j > 0 \rightarrow \text{not possible to reduce the cost function} \end{array}\right\} \textit{optimum}$$

Selection of a basic variable

- Determine the pivot row for the elimination process
- x_r: nonbasic variable $\rightarrow$ basic
  - If all $a_{i,r}$ are nonpositive in the r-th column, it is unbounded problem

always $b_i - a_{i,r} x_r \ge 0$
$\rightarrow$ no limit on x_r

$(r-th$ nonbasic column)

$$\left.\begin{aligned} x_1 + \cdots + a_{1,m+1}x_{m+1} + \cdots + a_{1,n}x_n &= b_1 - a_{1,r}x_r \\ x_2 + \cdots + a_{2,m+1}x_{m+1} + \cdots + a_{2,n}x_n &= b_2 - a_{2,r}x_r \\ &\vdots \\ x_m + a_{m,m+1}x_{m+1} + \cdots + a_{m,n}x_n &= b_m - a_{m,r}x_r \end{aligned}\right\}$$

$*$ new solution should be feasible $\rightarrow b_i - a_{i,r}x_r \ge 0$

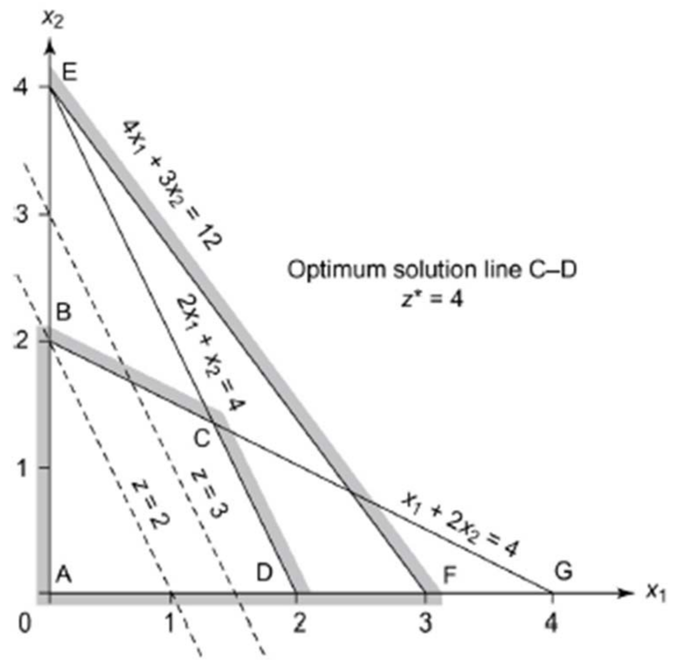
$$\min_i \left\{ \frac{b_i}{a_{i,r}},\ a_{i,r} > 0;\ i = 1, \ldots, m \right\} : \text{row with the smallest ratio}$$

Example 8.7

$$\begin{aligned} \text{Maximize} \quad & z = 2x_1 + x_2 \\ \text{subject to} \quad & 4x_1 + 3x_2 \le 12 \\ & 2x_1 + x_2 \le 4 \\ & x_1 + 2x_2 \le 4 \\ & x_1, x_2 \ge 0 \end{aligned}$$

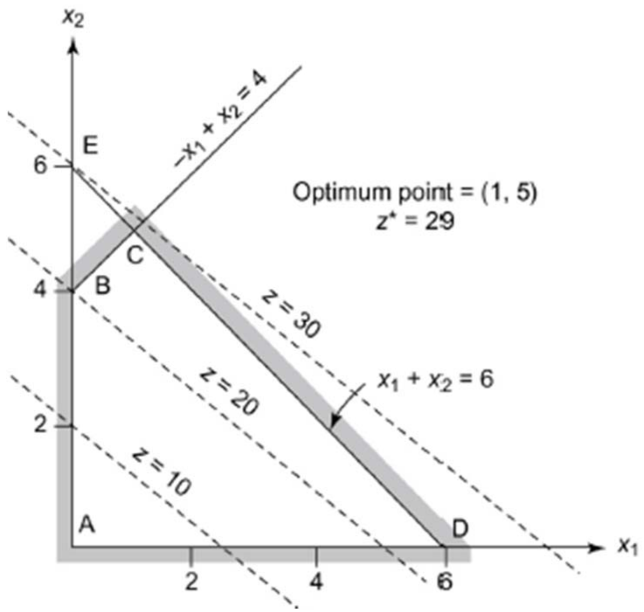
$\downarrow$

$$\left\{ \begin{aligned} \text{Minimize} \quad & f = -2x_1 - x_2 \\ \text{subject to} \quad & 4x_1 + 3x_2 + x_3 = 12 \\ & 2x_1 + x_2 + x_4 = 4 \\ & x_1 + 2x_2 + x_5 = 4 \\ & x_i \ge 0; \quad i = 1, \ldots, 5 \end{aligned} \right.$$



Example 8.8 ← 8.3

$$\left.\begin{aligned} &\textit{Maximize} \quad z = 4x_1 + 5x_2 \\ &\textit{subject to} \quad -x_1 + x_2 \le 4 \\ &\qquad\qquad x_1 + x_2 \le 6 \\ &\qquad\qquad x_1, x_2 \ge 0 \end{aligned}\right\} \rightarrow \left\{\begin{aligned} &\textit{Minimize} \quad f = -4x_1 - 5x_2 \\ &\textit{subject to} \quad -x_1 + x_2 + x_3 = 4 \\ &\qquad\qquad x_1 + x_2 + x_4 = 6 \\ &\qquad\qquad x_i \ge 0; \quad i = 1, \ldots, 4 \end{aligned}\right.$$



Multiple Solutions (Example 8.10)

(Sec 3.4)

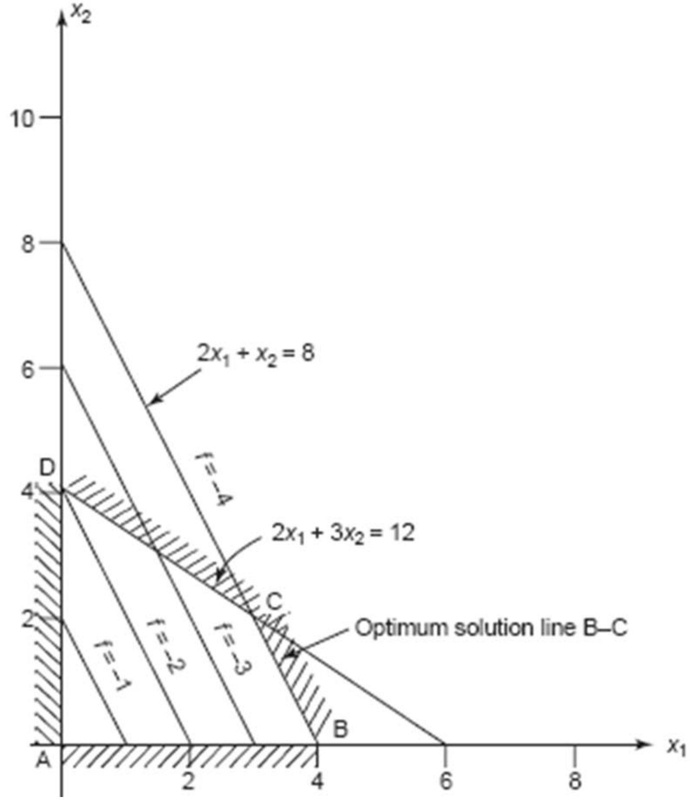
Minimize $f = -x_1 - 0.5x_2$

subject to $2x_1 + 3x_2 \le 12$

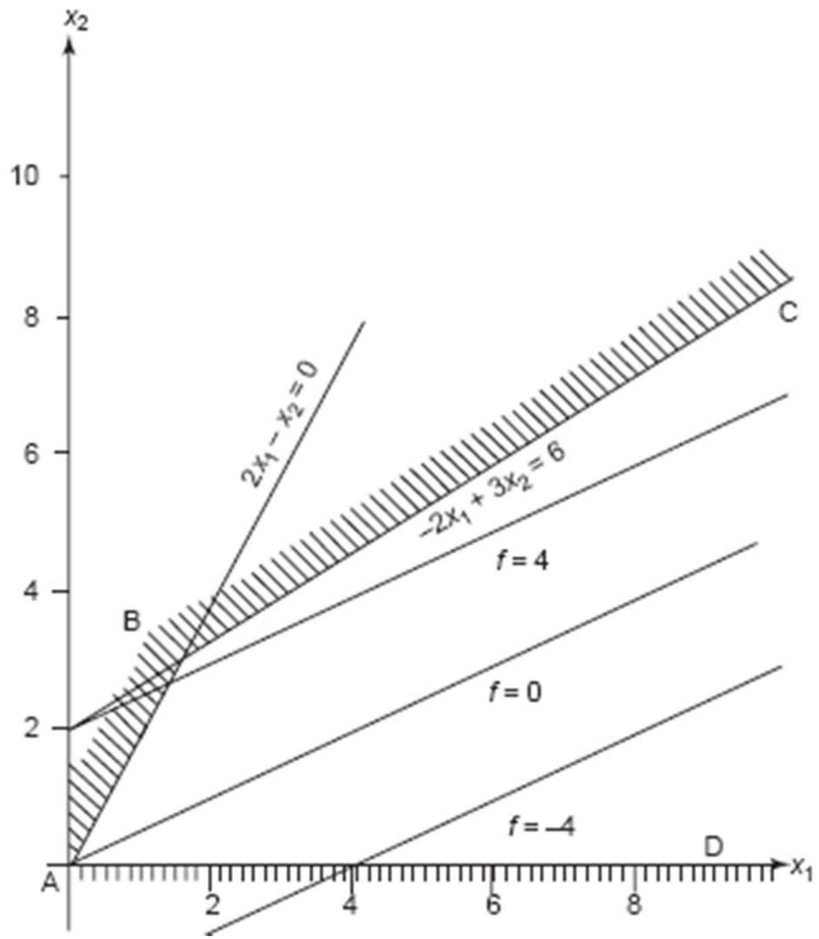
$2x_1 + x_2 \le 8$

If a reduced cost coefficient corresponding to a nonbasic variable is zero

	Basic ↓	x_1	x_2	x_3	x_4	b	
	x_3	2	3	1	0	12	*Initial tableau,*
x_4 out, x_1 in	$\rightarrow x_4$	$\underline{2}$	1	0	1	8	*pivot*: a_{21}
		$\underline{-1}$	-0.5	0	0	$f - 0$	
x_3 out, x_1 in	$\rightarrow x_3$	0	$\underline{2}$	1	-1	4	*Second tableau,*
	x_1	1	$\frac{1}{2}$	0	$\frac{1}{2}$	4	*pivot*: a_{12}
	Cost	0	$\underline{0}$	0	$\frac{1}{2}$	$f + 4^a$	
	x_2	0	1	$\frac{1}{2}$	$-\frac{1}{2}$	2	*Third tableau*
	x_1	1	0	$-\frac{1}{4}$	$\frac{3}{4}$	3	
	Cost	0	0	0 ↑	$\frac{1}{2}$ ↑	$f + 4^b$	



Unbounded Problem (Example 8.11)



(Sec 3.5)

$$\left.\begin{array}{ll}\text{Maximize} & z = x_1 - 2x_2 \\ \text{subject to} & 2x_1 - x_2 \ge 0 \\ & -2x_1 + 3x_2 \le 6 \\ & x_1, x_2 \ge 0\end{array}\right\} \rightarrow \left\{\begin{array}{ll}\text{Minimize} & f = -x_1 + 2x_2 \\ \text{subject to} & -2x_1 + x_2 \le 0 \\ & -2x_1 + 3x_2 \le 6 \\ & x_1, x_2 \ge 0\end{array}\right.$$

Basic ↓	x_1	x_2	x_3	x_4	**b**
1. x_3	−2	1	1	0	0
2. x_4	−2	3	0	1	6
3. Cost	−1	2	0	0	$f - 0$