Two-phase Simplex Method (1)

- How to handle "$\geq$ type" or "equality" constraints ?
  - Initial basic feasible solution: not available $\rightarrow$ artificial variables
  - New expanded space $\rightarrow$ original space
  - Artificial variables: basic $\rightarrow$ nonbasic
- To eliminate artificial variables
  - Artificial cost function: sum of all the artificial variables
- Phase I: to obtain the initial basic feasible solution

$$\begin{aligned} &\textit{Minimize} \quad w = \sum_{i=1}^{m} x_{n+i} \\ &\textit{subject to} \quad \sum_{j=1}^{n} a_{ij} x_j + x_{n+i} = b_i; \quad i = 1, \ldots, m \\ &\qquad\qquad\quad x_j \geq 0; \quad j = 1, \ldots, n \end{aligned}$$

Two-phase Simplex Method (2)

- Not suitable form for the Simplex method
  - Artificial cost function: expressed in terms of nonbasic variables

$$\underbrace{w = \sum_{i=1}^{m} x_{n+i}}_{\substack{\text{in terms of}\\ \text{basic variables}}} = \sum_{i=1}^{m}\left(b_i - \sum_{j=1}^{n} a_{ij}x_j\right) = \sum_{i=1}^{m} b_i - \sum_{j=1}^{n}\sum_{i=1}^{m} a_{ij}x_j = \underbrace{\sum_{i=1}^{m} b_i + \sum_{j=1}^{n} c'_j x_j}_{\substack{\text{in terms of}\\ \text{nonbasic variables}}}$$

$$c'_j = -\sum_{i=1}^{m} a_{ij}; \qquad j = 1, \ldots, n$$

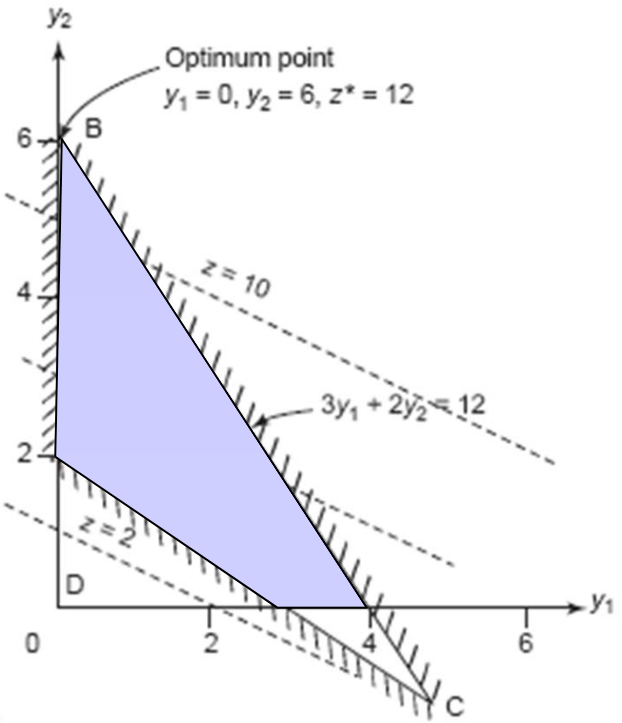
original cost function $\rightarrow$ treated as a constraint

$$\begin{cases} w = 0: \text{ end of Phase I} \\ w \neq 0: \text{ infeasible problem} \end{cases}$$

Example 8.14 ("≥ type")

$$\begin{aligned} &\text{Maximize} && z = y_1 + 2y_2 \\ &\text{subject to} && 3y_1 + 2y_2 \le 12 \\ & && 2y_1 + 3y_2 \ge 6 \\ & && y_1 \ge 0,\ y_2 \text{ unsigned} \end{aligned} \;\rightarrow\; \begin{cases} \text{Minimize} & f = -x_1 - 2x_2 + 2x_3 \\ \text{subject to} & 3x_1 + 2x_2 - 2x_3 + x_4 = 12 \\ & 2x_1 + 3x_2 - 3x_3 - x_5 + x_6 = 6 \\ & x_i \ge 0;\ \ i = 1, \ldots, 6 \end{cases}$$

$$w = x_6 = 6 - 2x_1 - 3x_2 + 3x_3 + x_5$$

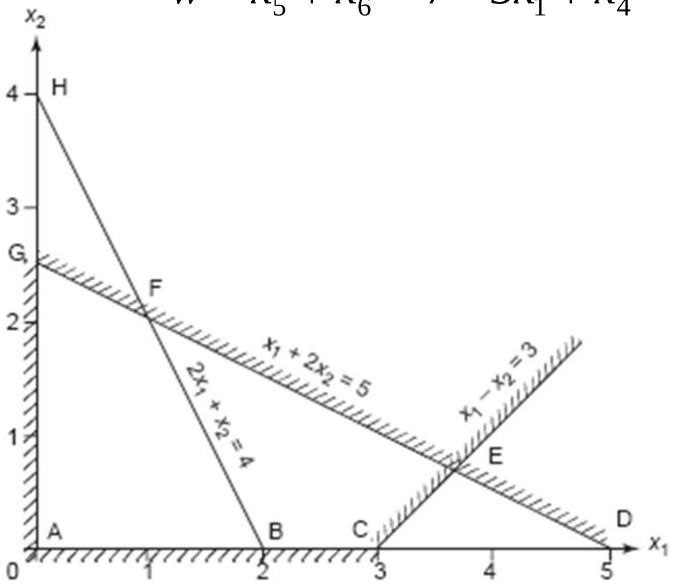


Basic ↓	x_1	x_2	x_3	x_4	x_5	x_6	b	
x_6 out, x_2 in — x_4	3	2	−2	1	0	0	12	*Phase I: initial tableau, pivot:* a_{22}
→ x_6	2	<u>3</u>	−3	0	−1	1	6	
Cost	−1	−2	2	0	0	0	$f - 0$	
Artificial cost	−2	<u>−3</u>	3	0	1	0	$w - 6$	
x_4 out, x_5 in — → x_4	$\frac{5}{3}$	0	0	1	$\underline{\frac{2}{3}}$	$-\frac{2}{3}$	8	*Second tableau pivot:* a_{15}
x_2	$\frac{2}{3}$	1	−1	0	$-\frac{1}{3}$	$\frac{1}{3}$	2	
Cost	$\frac{1}{3}$	0	0	0	$\underline{-\frac{2}{3}}$	$\frac{2}{3}$	$f + 4$	
Artificial cost	0	0	0	0	0	1	$w - 0$	*End of Phase I*
x_5	$\frac{5}{2}$	0	0	$\frac{3}{2}$	1	−1	12	*Third tableau: optimum solution*
x_2	$\frac{3}{2}$	1	−1	$\frac{1}{2}$	0	0	6	
Cost	2 ↑	0	0	1 ↑	0	0	$f + 12$	*End of Phase II*

Example 8.15 (“equality”)

$$
\begin{aligned}
&\textit{Maximize} && z = x_1 + 4x_2\\
&\textit{subject to} && x_1 + 2x_2 \le 5\\
& && 2x_1 + x_2 = 4\\
& && x_1 - x_2 \ge 3\\
& && x_1, x_2 \ge 0
\end{aligned}
\quad\rightarrow\quad
\begin{aligned}
&\textit{Minimize} && f = -x_1 - 4x_2\\
&\textit{subject to} && x_1 + 2x_2 + x_3 = 5\\
& && 2x_1 + x_2 + x_5 = 4\\
& && x_1 - x_2 - x_4 + x_6 = 3\\
& && x_i \ge 0; \quad i = 1,\ldots,6
\end{aligned}
$$

$$w = x_5 + x_6 = 7 - 3x_1 + x_4$$



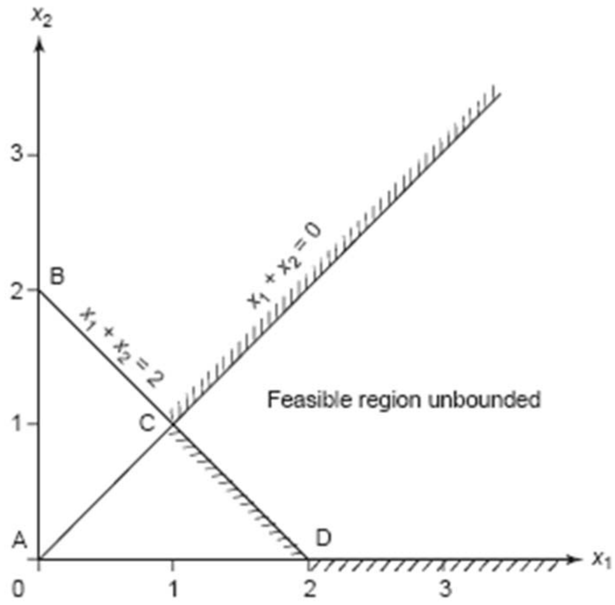
(infeasible problem)

	Basic ↓	x_1	x_2	x_3	x_4	x_5	x_6	b	
	x_3	1	2	1	0	0	0	5	*Initial tableau*
x_5 out, x_1 in	$\rightarrow x_5$	$\underline{2}$	1	0	0	1	0	4	*pivot: a_{21}*
	x_6	1	−1	0	−1	0	1	3	
	Cost	−1	−4	0	0	0	0	$f-0$	
Artificial cost		$\underline{-3}$	0	0	1	0	0	$w-7$	
	x_3	0	$\frac{3}{2}$	1	0	$-\frac{1}{2}$	0	3	*Second tableau*
	x_1	1	$\frac{1}{2}$	0	0	$\frac{1}{2}$	0	2	
	x_6	0	$-\frac{3}{2}$	0	−1	$-\frac{1}{2}$	1	1	
	Cost	0	$-\frac{7}{2}$	0	0	$\frac{1}{2}$	0	$f+2$	
Artificial cost		0	$\frac{3}{2}$	0	1	$\frac{3}{2}$	0	$w+1$	*End of Phase I*

Example 8.16 (unbounded)

$$\left.\begin{array}{ll} Maximize & z = 3x_1 - 2x_2 \\ subject\ to & x_1 - x_2 \geq 0 \\ & x_1 + x_2 \geq 2 \\ & x_1, x_2 \geq 0 \end{array}\right\} \rightarrow \left\{\begin{array}{ll} Minimize & f = -3x_1 + 2x_2 \\ subject\ to & -x_1 + x_2 + x_3 = 0 \\ & x_1 + x_2 - x_4 + x_5 = 2 \\ & x_i \geq 0; \quad i = 1, \ldots, 5 \end{array}\right.$$

$$w = x_5 = 2 - x_1 - x_2 + x_4$$



Basic ↓	x_1	x_2	x_3	x_4	x_5	b	
x_3	−1	1	1	0	0	0	*Initial tableau, pivot: a_{21}*
x_5 out, x_1 in → x_5	**<u>1</u>**	1	0	−1	1	2	
Cost	−3	2	0	0	0	$f - 0$	
Artificial cost	**<u>−1</u>**	−1	0	1	0	$w - 2$	
x_3	0	2	1	−1	1	2	*Second tableau*
x_1	1	1	0	−1	1	2	
Cost	0	5	0	**<u>−3</u>**	3	$f + 6$	
Artificial cost	0	0	0	0	1	$w - 0$	

End of Phase I *End of Phase II*

Example 8.17

- Degenerate basic feasible solution

$$
\begin{aligned}
&\textit{Maximize} && z = x_1 + 4x_2 \\
&\textit{subject to} && x_1 + 2x_2 \le 5 \\
& && 2x_1 + x_2 \le 4 \\
& && 2x_1 + x_2 \ge 4 \\
& && x_1 - x_2 \ge 1 \\
& && x_1, x_2 \ge 0
\end{aligned}
\quad \rightarrow
$$

$$
\begin{cases}
\textit{Minimize} & f = -x_1 - 4x_2 \\
\textit{subject to} & x_1 + 2x_2 + x_3 = 5 \\
& 2x_1 + x_2 + x_4 = 4 \\
& 2x_1 + x_2 - x_5 + x_7 = 4 \\
& x_1 - x_2 - x_6 + x_8 = 1 \\
& x_i \ge 0; \quad i = 1, \ldots, 8
\end{cases}
$$

$$w = x_7 + x_8 = 5 - 3x_1 + x_5 + x_6$$

Solution for Example 4.14 (degenerate basic feasible solution)

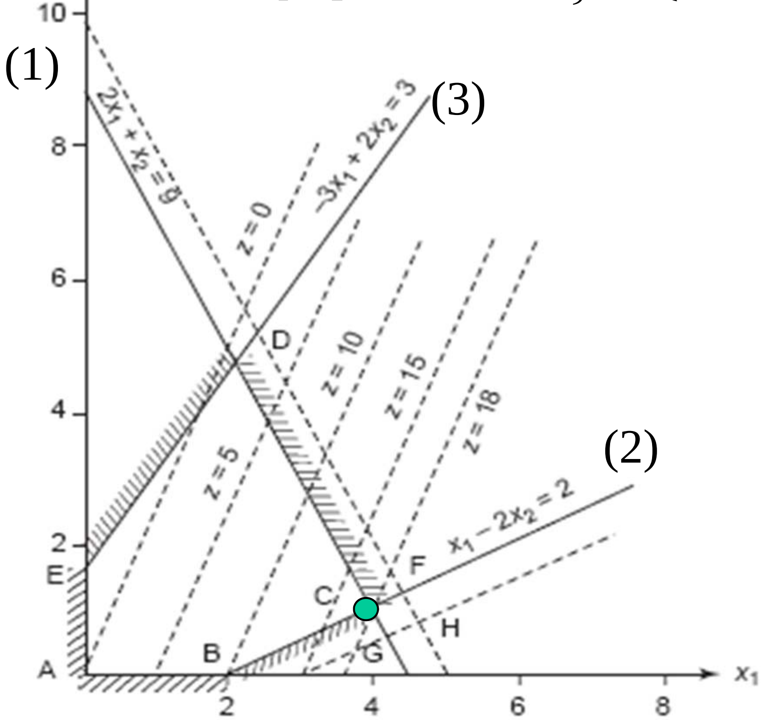
	Basic ↓	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	b	
	x_3	1	2	1	0	0	0	0	0	5	*Initial tableau, pivot: a_{14}*
	x_4	2	1	0	1	0	0	0	0	4	
	x_7	2	1	0	0	−1	0	1	0	4	
x_8 out, x_1 in	$\rightarrow x_8$	<u>1</u>	−1	0	0	0	−1	0	1	1	
	Cost	−1	−4	0	0	0	0	0	0	$f-0$	
	Artificial	<u>−3</u>	0	0	0	1	1	0	0	$w-5$	
	x_3	0	3	1	0	0	1	0	−1	4	*Second tableau, pivot: a_{23}*
	x_4	0	3	0	1	0	2	0	−2	2	
x_7 out, x_2 in	$\rightarrow x_7$	0	<u>3</u>	0	0	−1	2	1	−2	2	
	x_1	1	−1	0	0	0	−1	0	1	1	
	Cost	0	−5	0	0	0	−1	0	1	$f+1$	
	Artificial	0	<u>−3</u>	0	0	1	−2	0	3	$w-2$	
	x_3	0	0	1	0	1	−1	−1	1	2	*Third tableau, pivot: a_{25}*
x_4 out, x_5 in	$\rightarrow x_4$	0	0	0	1	<u>1</u>	0	−1	$-\frac{2}{3}$	0	
	x_2	0	1	0	0	$-\frac{1}{3}$	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	
	x_1	1	0	0	0	$-\frac{1}{3}$	$-\frac{1}{3}$	$\frac{1}{3}$	1	$\frac{5}{3}$	
	Cost	0	0	0	0	<u>$-\frac{5}{3}$</u>	$\frac{7}{3}$	$\frac{5}{3}$	1	$f+\frac{13}{3}$	
	Artificial	0	0	0	0	0	0	1	3	$w-0$	
	End of Phase I										
	x_3	0	0	1	−1	0	−1	0	1	2	*Final tableau*
	x_5	0	0	0	1	1	0	−1	0	0	
	x_2	0	1	0	$\frac{1}{3}$	0	$\frac{2}{3}$	0	$-\frac{2}{3}$	$\frac{2}{3}$	
	x_1	1	0	0	$\frac{1}{3}$	0	$-\frac{1}{3}$	0	$\frac{1}{3}$	$\frac{5}{3}$	
	Cost	0	0	0	$\frac{5}{3}$	0	$\frac{7}{3}$	0	$-\frac{7}{3}$	$f+\frac{13}{3}$	
	End of Phase II										

Post-Optimality Analysis

- Parameter changes: ***c***, ***b***, ***A*** $\rightarrow$ optimum solution ?
  - Use of the optimum solution of the original problem if changes in the parameters are within certain limits
  - Need information on Lagrange multiplier
- Lagrange multiplier for the *i*-th constraint
  - Reduced cost coefficient in the slack or artificial variable column associated w/ the *i*-th constraint
    - "$\leq$ type" : nonnegative (slack variable column)
    - "= type" : unrestricted in sign (artificial variable column)
    - "$\geq$ type" : nonpositive (artificial variable column = -surplus variable column)
  - $\frac{\partial f}{\partial e_i} = -y_i$: minimization problem w/ "$\leq$ type" or equality constraints

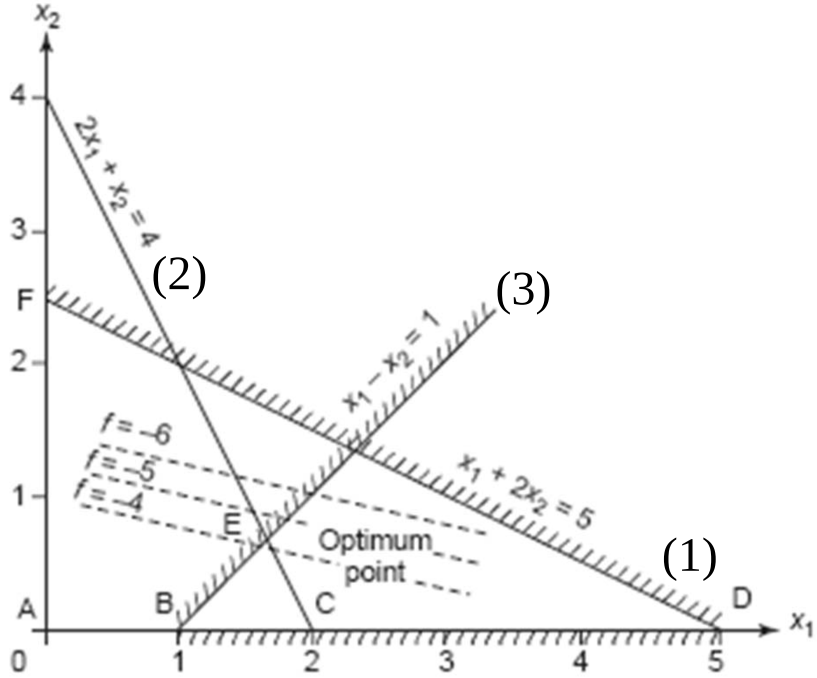
Example 8.18

$$\left.\begin{aligned} &\textit{Maximize} \quad z = 5x_1 - 2x_2 \\ &\textit{subject to} \quad (1)\ 2x_1 + x_2 \le 9 \\ &\qquad (2)\ x_1 - 2x_2 \le 2 \\ &\qquad (3)\ -3x_1 + 2x_2 \le 3 \\ &\qquad x_1, x_2 \ge 0 \end{aligned}\right\} \rightarrow \begin{cases} \dfrac{\partial f}{\partial e_1} = \qquad , \dfrac{\partial f}{\partial e_2} = \qquad , \dfrac{\partial f}{\partial e_3} = \\ (1)\ 9 \rightarrow 10, \\ (2)\ 2 \rightarrow 3, \\ (1)\ 9 \rightarrow 10\ \&\ (2)\ 2 \rightarrow 3, \end{cases}$$



Example 8.19 ← 8.13

$$
\left.\begin{array}{ll}
Maximize & z = x_1 + 4x_2 \\
subject\ to & (1)\ x_1 + 2x_2 \le 5 \\
 & (2)\ 2x_1 + x_2 = 4 \\
 & (3)\ x_1 - x_2 \ge 1 \\
 & x_1, x_2 \ge 0
\end{array}\right\} \rightarrow
\left\{\begin{array}{ll}
Minimize & f = -x_1 - 4x_2 \\
subject\ to & x_1 + 2x_2 + x_3 = 5 \\
 & 2x_1 + x_2 + x_5 = 4 \\
 & x_1 - x_2 - x_4 + x_6 = 1 \\
 & x_i \ge 0; \quad i = 1, \ldots, 6
\end{array}\right.
$$



$$
\begin{cases}
\dfrac{\partial f}{\partial e_2} = -y_2 = \quad ; \ (2)\ 4 \rightarrow 5, \ \ f = \\[2ex]
\dfrac{\partial f}{\partial e_3} = -y_3 = \quad ; \ (3)\ 1 \rightarrow 2, \ \ f =
\end{cases}
$$

Ranging

- Limits on changes in resources

$$\Delta_k : \text{possible change in } b_k$$

$$\max_i \left\{ \frac{-b_i'}{a_{ij}'} ; a_{ij}' > 0 \right\} \le \Delta_k \le \min_i \left\{ \frac{-b_i'}{a_{ij}'} ; a_{ij}' < 0 \right\}$$

$$b_i'' = b_i' + \Delta_k a_{ij}' : \text{new values of design variables}$$

- Ranging cost coefficients

$$c_k \to c_k + \Delta c_k$$

for cost coefficient of basic variables

$$\max_j \left\{ \frac{c_j'}{a_{rj}'} < 0 \right\} \le \Delta c_k \le \min_j \left\{ \frac{c_j'}{a_{rj}'} > 0 \right\}$$

r : row that determines x_k^*, j : nonbasic columns excluding artificial columns

$$f^*_{new} = f^* + \Delta c_k x_k^*$$

Example 8.18+20+22 (≤ type)

	Basic ↓	x_1	x_2	x_3	x_4	x_5	b
	x_3	2	1	1	0	0	9
x_4 out x_1 in	$\rightarrow x_4$	**1**	−2	0	1	0	2
		−3	2	0	0	1	3
	Cost	**−5**	2	0	0	0	$f-0$
x_3 out x_2 in	$\rightarrow x_3$	0	**5**	1	−2	0	5
	x_1	1	−2	0	1	0	2
		0	−4	0	3	1	9
	Cost	0	**−8**	0	5	0	$f+10$
	x_2	0	1	0.2	−0.4	0	1
	x_1	1	0	0.4	0.2	0	4
	x_5	0	0	0.8	1.4	1	13
	Cost	0	0	1.6	1.8	0	$f+18$
		↑ c_1'	↑ c_2'	↑ c_3'	↑ c_4'	↑ c_5'	

$$[\text{constraint } 1 \rightarrow x_3(\text{column } 3)]$$

$$r_i = -\frac{b_i'}{a_{i3}'} = \left\{-\frac{1}{0.2}, -\frac{4}{0.4}, -\frac{13}{0.8}\right\} = \{-5.0, -10.0, -16.25\}$$

$$-5.0 \le \Delta_1 \le \infty \xrightarrow{b_1=9} 4 \le b_1 \le \infty$$

$$[\text{cost coefficient } 1 \rightarrow x_1(\text{row } 2)]$$

$$d_j = \frac{c_j'}{a_{2j}'} = \left\{\frac{1.6}{0.4}, \frac{1.8}{0.2}\right\} = \{4.0, 9.0\}$$

$$-\infty \le \Delta c_1 \le 4 \xrightarrow{c_1=-5} -\infty \le c_1 \le -1$$

$$[\text{constraint } 1: 9 \rightarrow 10(\Delta_1 = 1)]$$

$$\begin{cases} x_2 = b_1'' = b_1' + \Delta_1 a_{13}' = 1 + (1)(0.2) = 1.2 \\ x_1 = b_2'' = b_2' + \Delta_1 a_{23}' = 4 + (1)(0.4) = 4.4 \\ x_5 = b_3'' = b_3' + \Delta_1 a_{33}' = 13 + (1)(0.8) = 13.8 \end{cases}$$

$$[\text{cost coefficient } 1: -5 \rightarrow -4(\Delta c_1 = 1)]$$

$$f_{new}^* = f^* + \Delta c_1 x_1^* = -18 + (1)(4) = -14$$

Example 8.19+21+23 (= or ≥ type)

Basic ↓	x_1	x_2	x_3	x_4	x_5	x_6	b
x_3	1	2	1	0	0	0	5
x_5	2	1	0	0	1	0	4
x_6	$\underline{1}$	−1	0	−1	0	1	1
Cost	−1	−4	0	0	0	0	$f-0$
Artificial	$\underline{-3}$	0	0	1	0	0	$w-5$
x_3	0	3	1	1	0	−1	4
x_5	0	$\underline{3}$	0	2	1	−2	2
x_1	1	−1	0	−1	0	1	1
Cost	0	−5	0	−1	0	1	$f+1$
Artificial	0	$\underline{-3}$	0	−2	0	3	$w-2$
x_3	0	0	1	−1	−1	1	2
x_2	0	1	0	$\frac{2}{3}$	$\frac{1}{3}$	$-\frac{2}{3}$	$\frac{2}{3}$
x_1	1	0	0	$-\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{5}{3}$
Cost	0 (c'_1)	0 (c'_2)	0 (c'_3)	$\frac{7}{3}$ (c'_4)	$\frac{5}{3}$ (c'_5)	$-\frac{7}{3}$ (c'_6)	$f+\frac{13}{3}$
Artificial	0	0	0	0	1	1	$w-0$
	End of Phase I				*End of Phase II*		

[constraint $2 \rightarrow x_5$ (column 5)]

$$r_i = -\frac{b'_i}{a'_{i5}} = \left\{-\frac{2}{-1}, -\frac{2/3}{1/3}, -\frac{5/3}{1/3}\right\} = \{2.0, -2.0, -5.0\}$$

$$-2.0 \le \Delta_2 \le 2.0 \xrightarrow{b_2=4} 2.0 \le b_2 \le 6.0$$

[cost coefficient $1 \rightarrow x_1$ (row 3), exclude artificial columns]

$$d_j = \frac{c'_j}{a'_{3j}} = \left\{\frac{7/3}{-1/3}\right\} = \{-7.0\}$$

$$-7.0 \le \Delta c_1 \le \infty \xrightarrow{c_1=-1} -8.0 \le c_1 \le \infty$$

[constraint $2: 4 \rightarrow 5 (\Delta_2 = 1)$]

$$\begin{cases} x_3 = b''_1 = b'_1 + \Delta_2 a'_{15} = 2 + (1)(-1) = 1.0 \\ x_2 = b''_2 = b'_2 + \Delta_2 a'_{25} = 2/3 + (1)(1/3) = 1.0 \\ x_1 = b''_3 = b'_3 + \Delta_2 a'_{35} = 5/3 + (1)(1/3) = 2.0 \end{cases}$$

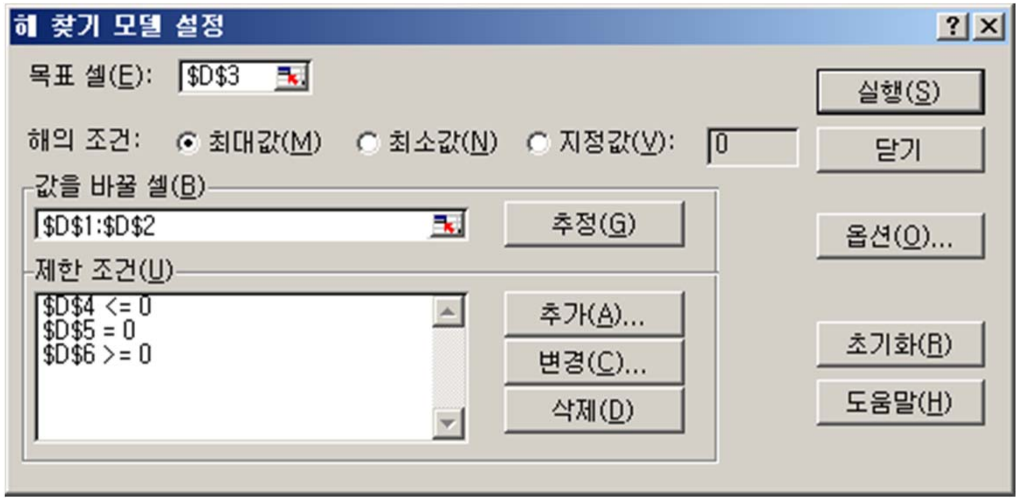
[cost coefficient $1: -1 \rightarrow -2 (\Delta c_1 = -1)$]

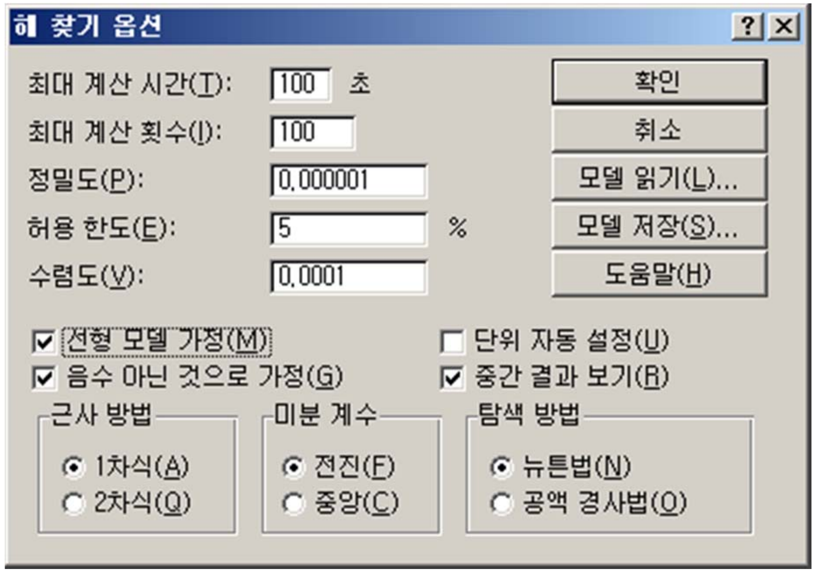
$$f^*_{new} = f^* + \Delta c_1 x^*_1 = -\frac{13}{3} + (-1)\left(\frac{5}{3}\right) = -6$$

LP in Excel Solver: Example 8.19

	A	B	C	D
1	x1	0		1.66667
2	x2	0		0.66667
3	z	0	max	4.33333
4	g1	-5	<= 0	-2
5	g2	-4	= 0	0
6	g3	-1	>= 0	0

1	2
1	1.666666667
0	0.666666667
1	4.333333333





Reports : Example 8.19+21+23

민감도보고서

Range of cost coeff.

값을 바꿀 셀

셀	이름	계산 값	한계 비용	목표 셀 계수	허용 가능 증가치	허용 가능 감소치
D1	x1	1.666666667	0	1	7	1E+30
D2	x2	0.666666667	0	4	1E+30	3.5

$$-7.0 \le \Delta c_1 \le \infty \xrightarrow{c_1=-1} -8.0 \le c_1 \le \infty$$

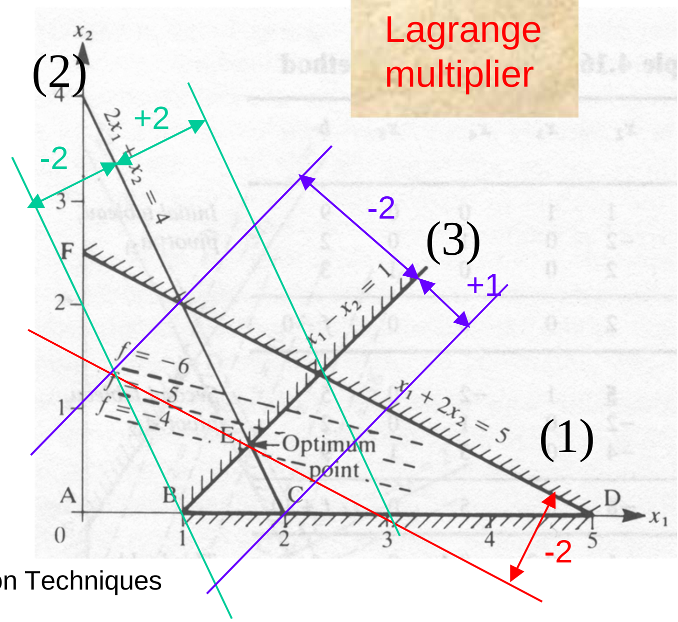
$$-\infty \le \Delta c_2 \le 3.5 \xrightarrow{c_2=-4} -\infty \le c_2 \le -0.5$$

제한 조건

셀	이름	계산 값	제한 조건 우변	허용 가능 증가치	허용 가능 감소치
D6	>= 0	0	0	1	2
D4	<= 0	−2	0	1E+30	2
D5	0	0	0	2	2

Lagrange multiplier

Range of resource limit



$$(1)\ x_1 + 2x_2 \le 5 \rightarrow -2 \le \Delta_1 \le \infty \rightarrow 3 \le b_1 \le \infty$$

$$(2)\ 2x_1 + x_2 = 4 \rightarrow -2 \le \Delta_2 \le 2 \rightarrow 2 \le b_2 \le 6$$

$$(3)\ x_1 - x_2 \ge 1 \rightarrow -2 \le \Delta_3 \le 1 \rightarrow -1 \le b_3 \le 2$$

Reports : Example 6.18+20+22

목표 셀 (최대값)

셀	이름	계산 전의 값	계산 값
D4	max optimal	0	18

값을 바꿀 셀

셀	이름	계산 전의 값	계산 값
D2	x1 optimal	0	4
D3	x2 optimal	0	1

제한 조건

셀	이름	셀의 값	수식	만족 정도	조건과의 차
D5	<=0 optimal	3.40721E-10	D5<=0	만족	0
D7	<=0 optimal	-13	D7<=0	부분적 만족	13
D6	<=0 optimal	-1.3102E-11	D6<=0	만족	0

	목표 셀	
셀	이름	값
D4	max optimal	18

	값을 바꿀 셀		하한값	목표 셀	상한값	목표 셀
셀	이름	값	한계값	결과	한계값	결과
D2	x1 optimal	4	0	-2	4	18
D3	x2 optimal	1	1	18	1	18

값을 바꿀 셀

셀	이름	계산 값	한계 비용	목표 셀 계수	허용 가능 증가치	허용 가능 감소치
D2	x1 optimal	4	0	5	1E+30	4
D3	x2 optimal	1	0	-2	4.5	8

제한 조건

셀	이름	계산 값	잠재 가격	제한 조건 우변	허용 가능 증가치	허용 가능 감소치
D5	<=0 optimal	3.40721E-10	1.6	0	1E+30	5
D7	<=0 optimal	-13	0	0	1E+30	13
D6	<=0 optimal	-1.3102E-11	1.8	0	2.5	9.285714286

Matlab Optimization Toolbox

```
>> [x,fval,exitflag,output,lambda] = linprog([-1 -4],[1 2;-1 1],[5;-1],[2 1],[4])

x =
    1.6667
    0.6667

fval =
   -4.3333

exitflag =
     1

output =
      iterations: 5
       algorithm: 'large-scale: interior point'
    cgiterations: 0
         message: 'Optimization terminated.'

lambda =

    ineqlin: [2x1 double]
      eqlin: 1.6667
      upper: [2x1 double]
      lower: [2x1 double]
```

```
>> lambda.ineqlin

ans =

    0.0000
    2.3333
```