Revised Simplex Method (1)

- Conventional simplex approach
  - Unnecessary bookkeeping, computationally inefficient
- Revised simplex method: more efficient
  - Necessary elements only (***B***$^{-1}$, ***y***, ***b***')

$$\left.\begin{aligned} &\text{Maximize} \quad \mathbf{c}^T\mathbf{x} \\ &\text{subject to} \quad \mathbf{A}\mathbf{x}=\mathbf{b} \\ &\qquad\qquad \mathbf{x}\ge\mathbf{0} \end{aligned}\right\} \xrightarrow[\mathbf{c}=[\mathbf{c}_B \;\; \mathbf{c}_N]]{\mathbf{x}=[\mathbf{x}_B \;\; \mathbf{x}_N],\ \mathbf{A}=[\mathbf{B} \;\; \mathbf{N}]} \left\{\begin{aligned} &\text{Maximize} \quad \mathbf{c}_B{}^T\mathbf{x}_B+\mathbf{c}_N{}^T\mathbf{x}_N \\ &\text{subject to} \quad \mathbf{A}\mathbf{x}=[\mathbf{B} \quad \mathbf{N}]\begin{bmatrix}\mathbf{x}_B \\ \mathbf{x}_N\end{bmatrix}=\mathbf{B}\mathbf{x}_B+\mathbf{N}\mathbf{x}_N=\mathbf{b} \\ &\qquad\qquad \mathbf{x}\ge\mathbf{0} \end{aligned}\right.$$

$$\mathbf{B}\mathbf{x}_B+\mathbf{N}\mathbf{x}_N=\mathbf{b} \rightarrow \mathbf{x}_B=\mathbf{B}^{-1}\mathbf{b}-\mathbf{B}^{-1}\mathbf{N}\mathbf{x}_N$$

$$\rightarrow \mathbf{c}_B{}^T\mathbf{x}_B+\mathbf{c}_N{}^T\mathbf{x}_N=\mathbf{c}_B{}^T\left(\mathbf{B}^{-1}\mathbf{b}-\mathbf{B}^{-1}\mathbf{N}\mathbf{x}_N\right)+\mathbf{c}_N{}^T\mathbf{x}_N=\mathbf{c}_B{}^T\mathbf{B}^{-1}\mathbf{b}-\left(\underbrace{\mathbf{c}_B{}^T\mathbf{B}^{-1}\mathbf{N}-\mathbf{c}_N{}^T}_{\mathbf{r}_N{}^T}\right)\mathbf{x}_N$$

$$\mathbf{r}_N{}^T=\mathbf{c}_B{}^T\mathbf{B}^{-1}\mathbf{N}-\mathbf{c}_N{}^T \xrightarrow{\mathbf{c}_B{}^T\mathbf{B}^{-1}=\pi^T \rightarrow \mathbf{B}^T\pi=\mathbf{c}_B} \mathbf{r}_N{}^T=\pi^T\mathbf{N}-\mathbf{c}_N{}^T$$

(reduced costs of the nonbasic variables $\mathbf{x}_N$)

Revised Simplex Method (2)

– Evaluation of reduced coefficients of the objective function

$$\begin{cases} \boldsymbol{r}_N{}^T = \boldsymbol{c}_B{}^T \boldsymbol{B}^{-1} \boldsymbol{N} - \boldsymbol{c}_N{}^T \rightarrow \text{pivot column } j \\ \boldsymbol{y} = \boldsymbol{B}^{-1}\left[\boldsymbol{A}_j\right], \boldsymbol{b}' = \boldsymbol{B}^{-1}\boldsymbol{b} \rightarrow \text{pivot row } i \end{cases}$$

$\boldsymbol{B}_k$: known

$\boldsymbol{B}_{k+1}$: one column of $\boldsymbol{B}_k$ is replaced

$$\boldsymbol{B}_k{}^{-1}\boldsymbol{B}_{k+1} = \begin{bmatrix} 1 & 0 & \cdots & y_1 & \cdots & 0 \\ 0 & 1 & \cdots & y_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & & & \vdots \\ 0 & 0 & & y_i & & 0 \\ \vdots & \vdots & & & \ddots & \vdots \\ 0 & 0 & \cdots & y_m & \cdots & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & \cdots & y_1 & \cdots & 0 \\ 0 & 1 & \cdots & y_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & & & \vdots \\ 0 & 0 & & y_i & & 0 \\ \vdots & \vdots & & & \ddots & \vdots \\ 0 & 0 & \cdots & y_m & \cdots & 1 \end{bmatrix} \boldsymbol{B}_{k+1}{}^{-1} = \boldsymbol{B}_k{}^{-1}$$

Duality in LP (1)

- Duality: primal ↔ dual
  - Electric potential ↔ Electric flow
  - Strain(Displacement) ↔ Stress(Force)
  - Price per unit of product ↔ Price per unit of resource

$$\left.\begin{array}{ll}\text{Maximize} & z_p = \mathbf{d}^T\mathbf{x} \\ \text{subject to} & \mathbf{A}\mathbf{x} \le \mathbf{e} \\ & \mathbf{x} \ge 0\end{array}\right\} \leftrightarrow \left\{\begin{array}{ll}\text{Minimize} & f_d = \mathbf{e}^T\mathbf{y} \\ \text{subject to} & \mathbf{A}^T\mathbf{y} \ge \mathbf{d} \\ & \mathbf{y} \ge 0\end{array}\right.$$

$$\left.\begin{array}{l} L(\mathbf{x},\mathbf{y}) = \mathbf{d}^T\mathbf{x} - \mathbf{y}^T(\mathbf{A}\mathbf{x}-\mathbf{e}) + \mathbf{u}^T\mathbf{x} \\ \dfrac{\partial L}{\partial \mathbf{x}} = \mathbf{0} \to \mathbf{A}^T\mathbf{y} - \mathbf{u} = \mathbf{d} \\ \mathbf{A}\mathbf{x} \le \mathbf{e} \\ \mathbf{x} \ge \mathbf{0} \\ \mathbf{y} \ge \mathbf{0}, \mathbf{y}^T(\mathbf{A}\mathbf{x}-\mathbf{e}) = \mathbf{0} \\ \mathbf{u} \ge \mathbf{0}, -\mathbf{u}^T\mathbf{x} = \mathbf{0} \end{array}\right\} \xleftrightarrow[-\mathbf{v}=\mathbf{A}\mathbf{x}-\mathbf{b}]{\mathbf{u}=\mathbf{A}^T\mathbf{y}-\mathbf{c}} \left\{\begin{array}{l} L(\mathbf{y},\mathbf{x}) = \mathbf{e}^T\mathbf{y} + \mathbf{x}^T(\mathbf{d}-\mathbf{A}^T\mathbf{y}) - \mathbf{v}^T\mathbf{y} \\ \dfrac{\partial L}{\partial \mathbf{y}} = \mathbf{0} \to \mathbf{A}\mathbf{x} + \mathbf{v} = \mathbf{b} \\ \mathbf{A}^T\mathbf{y} \ge \mathbf{c} \\ \mathbf{y} \ge \mathbf{0} \\ \mathbf{x} \ge \mathbf{0}, \mathbf{x}^T(\mathbf{d}-\mathbf{A}^T\mathbf{y}) = \mathbf{0} \\ \mathbf{y} \ge \mathbf{0}, -\mathbf{v}^T\mathbf{y} = \mathbf{0} \end{array}\right.$$

Examples

$$\left.\begin{aligned}&\text{Maximize} \quad z_p = 5x_1 - 2x_2\\&\text{subject to} \quad 2x_1 + x_2 \le 9\\&\qquad x_1 - 2x_2 \le 2\\&\qquad -3x_1 + 2x_2 \le 3\\&\qquad x_1,\ x_2 \ge 0\end{aligned}\right\} \leftrightarrow \left\{\begin{aligned}&\text{Minimize} \quad f_d = 9y_1 + 2y_2 + 3y_3\\&\text{subject to} \quad 2y_1 + y_2 - 3y_3 \ge 5\\&\qquad y_1 - 2y_2 + 2y_3 \ge -2\\&\qquad y_1,\ y_2 \ge 0\end{aligned}\right.$$

$$\left.\begin{aligned}&\text{Maximize} \quad z_p = x_1 + 4x_2\\&\text{subject to} \quad x_1 + 2x_2 \le 5\\&\qquad 2x_1 + x_2 = 4 \rightarrow \begin{cases}2x_1 + x_2 \le 4\\2x_1 + x_2 \ge 4 \rightarrow -2x_1 - x_2 \le -4\end{cases}\\&\qquad x_1 - x_2 \ge 1 \rightarrow -x_1 + x_2 \le -1\\&\qquad x_1,\ x_2 \ge 0\end{aligned}\right\}$$

$$\leftrightarrow \left\{\begin{aligned}&\text{Minimize} \quad f_d = 5y_1 + 4\left(y_2 - y_3\right) - y_4\\&\text{subject to} \quad y_1 + 2\left(y_2 - y_3\right) - y_4 \ge 1\\&\qquad 2y_1 + \left(y_2 - y_3\right) + y_4 \ge 4\\&\qquad y_1,\ y_2,\ y_3,\ y_4 \ge 0\end{aligned}\right. \leftrightarrow \left\{\begin{aligned}&\text{Minimize} \quad f_d = 5y_1 + 4y_5 - y_4\\&\text{subject to} \quad y_1 + 2y_5 - y_4 \ge 1\\&\qquad 2y_1 + y_5 + y_4 \ge 4\\&\qquad y_1,\ y_4 \ge 0,\ y_5 \left(= y_2 - y_3\right) \text{ unrestricted}\end{aligned}\right.$$

Equality constraint ↔ variable is unrestricted in sign

Duality in LP (2)

- The dual of the dual is the primal.
- Dual variables ***y*** are Lagrange multipliers for primal constraints.

$$
\begin{cases}
L(\mathbf{x},\mathbf{y}) = \mathbf{d}^T\mathbf{x} - \mathbf{y}^T(\mathbf{A}\mathbf{x}-\mathbf{b}) + \mathbf{u}^T\mathbf{x} \\
\dfrac{\partial L}{\partial \mathbf{x}} = \mathbf{0} \rightarrow \mathbf{A}^T\mathbf{y} - \mathbf{u} = \mathbf{d} \rightarrow -\mathbf{d} + \mathbf{A}^T\mathbf{y} = \mathbf{u} \rightarrow \begin{cases} \mathbf{u} > 0,\ \mathbf{x} = 0,\ \mathbf{A}^T\mathbf{y} > \mathbf{d} \\ \mathbf{u} = 0,\ \mathbf{x} > 0,\ \mathbf{A}^T\mathbf{y} = \mathbf{d} \end{cases} \\
\mathbf{A}\mathbf{x} \le \mathbf{e} \\
\mathbf{x} \ge \mathbf{0} \\
\mathbf{y} \ge \mathbf{0} \\
\mathbf{y}^T(\mathbf{A}\mathbf{x}-\mathbf{e}) = \mathbf{0} \rightarrow \mathbf{x}^T\mathbf{A}^T\mathbf{y} = \mathbf{y}^T\mathbf{e} \rightarrow \mathbf{d}^T\mathbf{x} = \mathbf{y}^T\mathbf{e} \leftarrow z_p \\
\mathbf{u} \ge \mathbf{0} \\
-\mathbf{u}^T\mathbf{x} = \mathbf{0}
\end{cases}
$$

Example

– Treatment of equality constraints

- Pair of inequalities / Unrestricted in sign

$$
\left.\begin{array}{ll}
<\text{Primal}> & \\
\textit{Maximize} & z_p = x_1 + 4x_2 \\
\textit{subject to} & x_1 + 2x_2 \le 5 \\
 & 2x_1 + x_2 = 4 \\
 & x_1 - x_2 \ge 1 \\
 & x_1, x_2 \ge 0
\end{array}\right\}
=
\left\{\begin{array}{ll}
<\text{Primal}> & \\
\textit{Maximize} & z_p = x_1 + 4x_2 \\
\textit{subject to} & x_1 + 2x_2 \le 5 \\
 & 2x_1 + x_2 \le 4 \\
 & -2x_1 - x_2 \le -4 \\
 & -x_1 + x_2 \le -1 \\
 & x_1, x_2 \ge 0
\end{array}\right.
\rightarrow
\left\{\begin{array}{ll}
<\text{Dual}> & \\
\textit{Minimize} & f_d = 5y_1 + 4(y_2 - y_3) - y_4 \\
\textit{subject to} & y_1 + 2(y_2 - y_3) - y_4 \ge 1 \\
 & 2y_1 + (y_2 - y_3) + y_4 \ge 4 \\
 & y_1, y_2, y_3, y_4 \ge 0
\end{array}\right.
$$

$$
\left.\begin{array}{ll}
<\text{Dual}> & \\
\textit{Minimize} & f_d = 5y_1 + 4y_5 - y_4 \\
\textit{subject to} & y_1 + 2y_5 - y_4 \ge 1 \\
 & 2y_1 + y_5 + y_4 \ge 4 \\
 & y_1, y_4 \ge 0 \\
 & y_5 = y_2 - y_3;\ \text{unsigned}
\end{array}\right\}
=
\left\{\begin{array}{ll}
<\text{Primal}> & \\
\textit{Maximize} & z_p = -5y_1 - 4y_5 + y_4 \\
\textit{subject to} & -y_1 - 2y_5 + y_4 \le -1 \\
 & -2y_1 - y_5 - y_4 \le -4 \\
 & y_1, y_4 \ge 0 \\
 & y_5;\ \text{unsigned}
\end{array}\right.
\rightarrow
\left\{\begin{array}{ll}
<\text{Dual of Dual}> & \\
\textit{Minimize} & f_d = -x_1 - 4x_2 \\
\textit{subject to} & -x_1 - 2x_2 \ge -5 \\
 & -2x_1 - x_2 = -4 \\
 & x_1 - x_2 \ge 1 \\
 & x_1, x_2 \ge 0
\end{array}\right.
$$

Duality in LP (3)

PRIMAL		DUAL
# of constraints (*i*-th)	=	# of variables (v_i)
# of variables (x_i)	=	# of constraints (*i*-th)
"$\leq$ Type" constraints	$\leftrightarrow$	"$\geq$ Type" constraints
Max cost function	$\leftrightarrow$	*Min* cost function
Coefficients c_i	$\leftrightarrow$	Right-hand side of constraints
Right-hand side of constraints b_i	$\leftrightarrow$	Coefficients of cost function
Coefficient [a_{ij}]	$\leftrightarrow$	Coefficient [a_{ji}]
Variable $x_i \geq 0$	$\leftrightarrow$	*i*-th constraint: inequality
Variable x_i unsigned	$\leftrightarrow$	*i*-th constraint: equality
i-th constraint: inequality	$\leftrightarrow$	Variable $v_i \geq 0$
i-th constraint: equality	$\leftrightarrow$	Variable v_i unsigned

Duality Theorem for LP

- If primal and dual problems have feasible solutions, then both have optimal solutions and the maximum of the primal objective is equal to the minimum of the dual objective.
- If either problem has an unbounded objective, the other problem is infeasible.

$$\left.\begin{array}{ll}\text{Maximize} & z_p = \mathbf{d}^T\mathbf{x} \\ \text{subject to} & \mathbf{A}\mathbf{x} \le \mathbf{e} \\ & \mathbf{x} \ge 0\end{array}\right\} \leftrightarrow \left\{\begin{array}{ll}\text{Minimize} & f_d = \mathbf{e}^T\mathbf{y} \\ \text{subject to} & \mathbf{A}^T\mathbf{y} \ge \mathbf{d} \\ & \mathbf{y} \ge 0\end{array}\right.$$

$$\Rightarrow \mathbf{d}^T\mathbf{x} \le \mathbf{y}^T\mathbf{A}\mathbf{x} \le \mathbf{y}^T\mathbf{e}$$

$\mathbf{x}_B$: basic solution at the optimum of the primal

$\mathbf{v}$: solution of the dual

$$\Rightarrow \mathbf{d}_B{}^T\mathbf{x}_B = \mathbf{y}^T\mathbf{e} \xrightarrow{\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{e} - \mathbf{B}^{-1}\mathbf{N}\mathbf{x}_N} \mathbf{y}^T\mathbf{e} = \mathbf{d}_B{}^T\mathbf{B}^{-1}\mathbf{e} \rightarrow \mathbf{y} = \mathbf{d}_B{}^T\mathbf{B}^{-1}$$

Primal $\leftrightarrow$ Dual

– If $z_p = f_d$, then ***x*** and ***y*** are solutions for primal and dual problems, respectively
  - ***x*** and ***y*** with $z_p = f_d$ maximize z_p while minimizing f_d

$$\mathbf{A}^T\mathbf{y} \geq \mathbf{d} \xrightarrow[\text{multiply } \mathbf{x}^T]{} \mathbf{x}^T\mathbf{A}^T\mathbf{y} \geq \mathbf{x}^T\mathbf{d} \rightarrow \mathbf{y}^T\mathbf{A}\mathbf{x} \geq \mathbf{d}^T\mathbf{x}$$

$$\xrightarrow[\mathbf{Ax} \leq \mathbf{e}]{} \mathbf{y}^T\mathbf{e} \geq \mathbf{d}^T\mathbf{x} \rightarrow f_d \geq z_p$$

– (primal is unbounded)$\leftrightarrow$(corresponding dual is infeasible)

– (*j*-th dual constraint: strict inequality) $\rightarrow$(corresponding *j*-th primal variable: nonbasic(=0))

$$\sum_{i=1}^{m} a_{ij} y_i > d_j \rightarrow x_j = 0$$

– (*i*-th dual variable: basic) $\rightarrow$ (*i*-th primal constraint: equality)

$$y_i > 0 \rightarrow \sum_{j=1}^{m} a_{ij} x_j = e_i$$

Dual $\rightarrow$ Primal

- (reduced cost coefficient of the slack or surplus variable associated w/ the *i*-th dual constraint) $\rightarrow$ (*i*-th primal variable)
- (reduced cost coefficient of nonbasic) $\rightarrow$ (value of the slack or surplus variable for the corresponding primal constraint)
- Dual variables $\rightarrow$("$\leq$" form only) $\rightarrow$ primal constraints
- Equality constraint ? (no slack or surplus variable)
  - Convert into a pair of inequalities
  - Artificial variable

Duality: Example (1)

< Primal >

$$\begin{aligned} &\textit{Maximize} \quad z_p = 5x_1 - 2x_2 \\ &\textit{subject to} \quad 2x_1 + x_2 \le 9 \\ &\qquad x_1 - 2x_2 \le 2 \\ &\qquad -3x_1 + 2x_2 \le 3 \\ &\qquad x_1, x_2 \ge 0 \end{aligned}$$

$\rightarrow$

< Dual >

$$\begin{aligned} &\textit{Minimize} \quad f_d = 9y_1 + 2y_2 + 3y_3 \\ &\textit{subject to} \quad 2y_1 + y_2 - 3y_3 \ge 5 \rightarrow 2y_1 + y_2 - 3y_3 - y_4 + y_6 = 5 \\ &\qquad y_1 - 2y_2 + 2y_3 \ge -2 \rightarrow -y_1 + 2y_2 - 2y_3 + y_5 = 2 \\ &\qquad y_1, y_2, y_3 \ge 0 \end{aligned}$$

Basic ↓	x_1	x_2	x_3	x_4	x_5	b
x_2	0	1	0.2	−0.4	0	1
x_1	1	0	0.4	0.2	0	4
x_5	0	0	0.8	1.4	1	13
Cost	0	0	1.6	1.8	0	$f_p + 18$

Basic ↓	y_1	y_2	y_3	y_4	y_5	y_6	b
y_6	2	1	−3	−1	0	1	5
y_5	−1	2	−2	0	1	0	2
Cost	9	2	3	0	0	0	$f_d - 0$
Artificial	−2	−1	3	1	0	0	$w - 5$
y_1	1	0.5	−1.5	−0.5	0	0.5	2.5
y_5	0	2.5	−3.5	−0.5	1	0.5	4.5
Cost	0	−2.5	16.5	4.5	0	−4.5	$f_d - 22.5$
Artificial	0	0	0	0	0	1	$w - 0$
y_1	1	0	−0.8	−0.4	−0.2	0.4	1.6
y_2	0	1	−1.4	−0.2	0.4	0.2	1.8
Cost	0	0	13.0	4.0	1.0	−4.0	$f_d - 18$

Duality: Example (2)

$$\left.\begin{array}{l} \text{< Primal >} \\ \text{basic}: x_1 = 4, x_2 = 1, x_5 = 13 \\ \text{nonbaic}: x_3 = x_4 = 0 \\ \text{maxium}: z_p = 18 \end{array}\right\} \leftrightarrow \left\{\begin{array}{l} \text{< Dual >} \\ \text{basic}: y_1 = 1.6, y_2 = 1.8 \\ \text{nonbaic}: y_3 = y_4 = y_5 = 0 \\ \text{minium}: f_d = 18 \end{array}\right.$$

(1) $f_d = z_p$

(2) y_1 & y_2 : *basic* $\rightarrow$ constraints 1 & 2 must be satisfied at equality (active)

$$\Rightarrow \left\{\begin{array}{l} 2x_1 + x_2 = 9 \\ x_1 - 2x_2 = 2 \end{array}\right\} \rightarrow x_1 = 4, x_2 = 1$$

(3) redued cost coeff. of y_4 (*surplus* : 1st constraint) $= 4.0 \rightarrow x_1$

redued cost coeff. of y_5 (*slack* : 2nd constraint) $= 1.0 \rightarrow x_2$

redued cost coeff. of y_3 (*nonbasic*) $= 13.0 \rightarrow$ (*slack* of 3rd constraint)

Duality: Example (3)

– We do NOT have to replace an equality constraint by two inequalities

$$
\begin{aligned}
&\textit{Maximize} \quad z = x_1 + 4x_2 \\
&\textit{subject to} \quad x_1 + 2x_2 \le 5 \\
&\qquad \left.\begin{aligned} 2x_1 + x_2 \le 4 \\ 2x_1 + x_2 \ge 4 \end{aligned}\right\} \to 2x_1 + x_2 = 4 \\
&\qquad x_1 - x_2 \ge 1 \\
&\qquad x_1, x_2 \ge 0
\end{aligned}
$$

(two inequalities)

Basic ↓	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	b
x_3	0	0	1	−1	0	−1	−0	−1	2
x_5	0	0	0	1	1	0	−1	0	0
x_2	0	1	0	$\frac{1}{3}$	0	$\frac{2}{3}$	0	$-\frac{2}{3}$	$\frac{2}{3}$
x_1	1	0	0	$\frac{1}{3}$	0	$-\frac{1}{3}$	0	$\frac{1}{3}$	$\frac{5}{3}$
Cost	0	0	0	$\frac{5}{3}$	0	$\frac{7}{3}$	0	$-\frac{7}{3}$	$f_p + \frac{13}{3}$
			y_1	y_2	y_3	y_4			

(equality)

Basic ↓	x_1	x_2	x_3	x_4	x_5	x_6	b
x_3	1	2	1	0	0	0	5
x_5	2	1	0	0	1	0	4
x_6	<u>1</u>	−1	0	−1	0	1	1
Cost	−1	−4	0	0	0	0	$f_p - 0$
Artificial	<u>−3</u>	0	0	1	0	0	$w - 5$
x_3	0	3	1	1	0	−1	4
x_5	0	<u>3</u>	0	2	1	−2	2
x_1	1	−1	0	−1	0	1	1
Cost	0	−5	0	−1	0	1	$f_p + 1$
Artificial	0	<u>−3</u>	0	−2	0	3	$w - 2$
x_3	0	0	1	−1	−1	1	2
x_2	0	1	0	$\frac{2}{3}$	$\frac{1}{3}$	$-\frac{2}{3}$	$\frac{2}{3}$
x_1	1	0	0	$-\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{5}{3}$
Cost	0	0	0	$\frac{7}{3}$	$\frac{5}{3}$	$-\frac{7}{3}$	$f_p + \frac{13}{3}$
Artificial	0	0	0	0	1	1	$w - 0$
			y_1	y_3	y_2		

End of Phase I   *End of Phase II*

Dual variable for equality constraint: $y_2 - y_3 = 5/3$

Quadratic Programming Problem

$$\left.\begin{aligned}&\text{Minimize}\quad q(\mathbf{x})=\mathbf{c}^T\mathbf{x}+\frac{1}{2}\mathbf{x}^T\mathbf{H}\mathbf{x}\\&\text{subject to}\quad \mathbf{N}^T\mathbf{x}=\mathbf{e}\\&\qquad\qquad \mathbf{A}^T\mathbf{x}\le\mathbf{b}\\&\qquad\qquad \mathbf{x}\ge 0\end{aligned}\right\}\rightarrow L=\mathbf{c}^T\mathbf{x}+\frac{1}{2}\mathbf{x}^T\mathbf{H}\mathbf{x}+\mathbf{u}^T\left(\mathbf{A}^T\mathbf{x}+\mathbf{s}-\mathbf{b}\right)-\boldsymbol{\xi}^T\mathbf{x}+\mathbf{v}^T\left(\mathbf{N}^T\mathbf{x}-\mathbf{e}\right)$$

$$\left.\begin{aligned}&\frac{\partial L}{\partial\mathbf{x}}=\mathbf{c}+\mathbf{H}\mathbf{x}+\mathbf{A}\mathbf{u}-\boldsymbol{\xi}+\mathbf{N}\mathbf{v}=\mathbf{0}\\&\frac{\partial L}{\partial\mathbf{u}}=\mathbf{A}^T\mathbf{x}+\mathbf{s}-\mathbf{b}=0\\&\frac{\partial L}{\partial\mathbf{v}}=\mathbf{N}^T\mathbf{x}-\mathbf{e}=0\\&u_is_i=0,\ u_i,s_i\ge 0,\ i=1,\ldots,m\\&\xi_ix_i=0,\ \xi_i,x_i\ge 0,\ i=1,\ldots,n\end{aligned}\right\}\xrightarrow{\mathbf{v}=\mathbf{y}-\mathbf{z}}\left\{\begin{aligned}&\begin{bmatrix}\mathbf{H}&\mathbf{A}&-\mathbf{I}&\mathbf{0}&\mathbf{N}&-\mathbf{N}\\\mathbf{N}^T&\mathbf{0}&\mathbf{0}&\mathbf{I}&\mathbf{0}&\mathbf{0}\\\mathbf{N}^T&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\end{bmatrix}\begin{bmatrix}\mathbf{x}\\\mathbf{u}\\\boldsymbol{\xi}\\\mathbf{s}\\\mathbf{y}\\\mathbf{z}\end{bmatrix}=\begin{bmatrix}-\mathbf{c}\\\mathbf{b}\\\mathbf{e}\end{bmatrix}\rightarrow\mathbf{B}\mathbf{X}=\mathbf{D}\\&X_iX_{n+m+i}=0,\ i=1,\ldots,(n+m)\\&X_i\ge 0,\ i=1,\ldots,2(n+m+p)\end{aligned}\right.$$

Simplex method for solving QP problem

- Wolfe(1959)→Hadley(1964)
  - Solve as a linear program using the Phase I simplex method
  - Fail to converge when Q is positive semidefinite
- Lemke(1965)
  - Complementary pivot method

$$\mathbf{BX} = \mathbf{D} \rightarrow \mathbf{BX} + \mathbf{Y} = \mathbf{D}$$

$$w = \sum_{i=1}^{n+m+p} Y_i = \sum_{i=1}^{n+m+p} D_i - \sum_{j=1}^{2(n+m+p)} \sum_{i=1}^{n+m+p} B_{ij} X_j = w_0 + \sum_{j=1}^{2(n+m+p)} C_j X_j$$

Karmarkar's Method

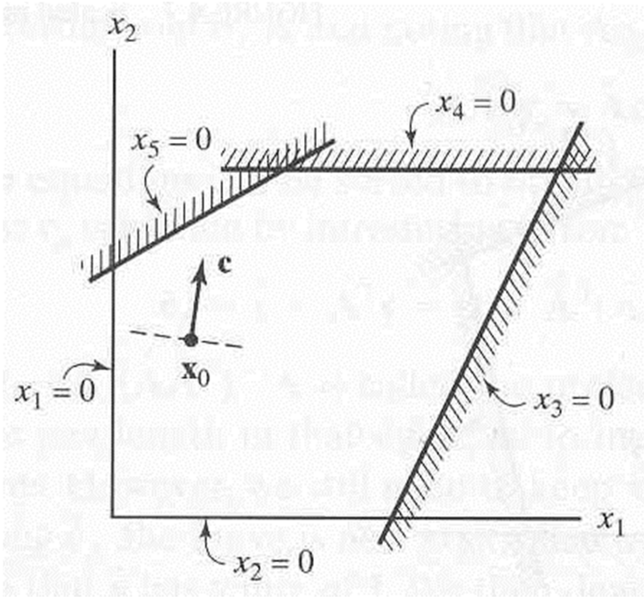
- Karmarkar, N., "A New Polynomial Time Algorithm for Linear Programming," *Combinatorica*, 4, 373-395, 1984

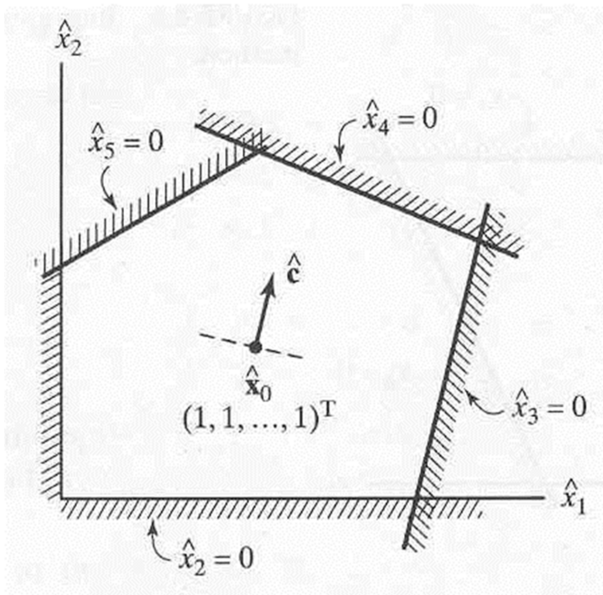
- Solve large scale LP problems very efficiently
  - 50 times faster than the simplex method
  - Karmarkar's method: 1 hr. for 150,000 DV's + 12,000 constraints (convergence: n^a)
  - Simplex method: 4 hrs. for 36,000 DV's + 10,000 constraints (convergence: β^n)
- Interior method
  - Find improve search directions in the interior of the feasible space
- Observations
  - If the current solution is near the center of the polytope, we can move along the steepest descent direction to reduce the cost by a maximum amount.
  - The solution space can always be transformed w/o changing the nature of the problem so that the current solution lies near the center of the polytope.

Interior Approach (1)

- special simplex structure (Karmarkar) $\rightarrow$ Affine scaling algorithm (Dikin, 1967) $\rightarrow$standard form

$$\left.\begin{array}{ll} \text{Maximize} & \boldsymbol{c}^T\boldsymbol{x} \\ \text{subject to} & \boldsymbol{A}\boldsymbol{x} \le \boldsymbol{b} \\ & \boldsymbol{x} \ge \boldsymbol{0} \end{array}\right\} \xrightarrow{\boldsymbol{D}=\begin{bmatrix} x_1^0 & 0 & \cdots & 0 \\ 0 & x_2^0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x_n^0 \end{bmatrix}, \boldsymbol{x}=\boldsymbol{D}\hat{\boldsymbol{x}}\ge\boldsymbol{0}, \hat{\boldsymbol{c}}=\boldsymbol{D}\boldsymbol{c}, \hat{\boldsymbol{A}}=\boldsymbol{A}\boldsymbol{D}} \left\{\begin{array}{ll} \text{Maximize} & \hat{\boldsymbol{c}}^T\hat{\boldsymbol{x}} \\ \text{subject to} & \hat{\boldsymbol{A}}\hat{\boldsymbol{x}} \le \hat{\boldsymbol{b}} \\ & \hat{\boldsymbol{x}} \ge \boldsymbol{0} \end{array}\right.$$





Interior Approach (2)

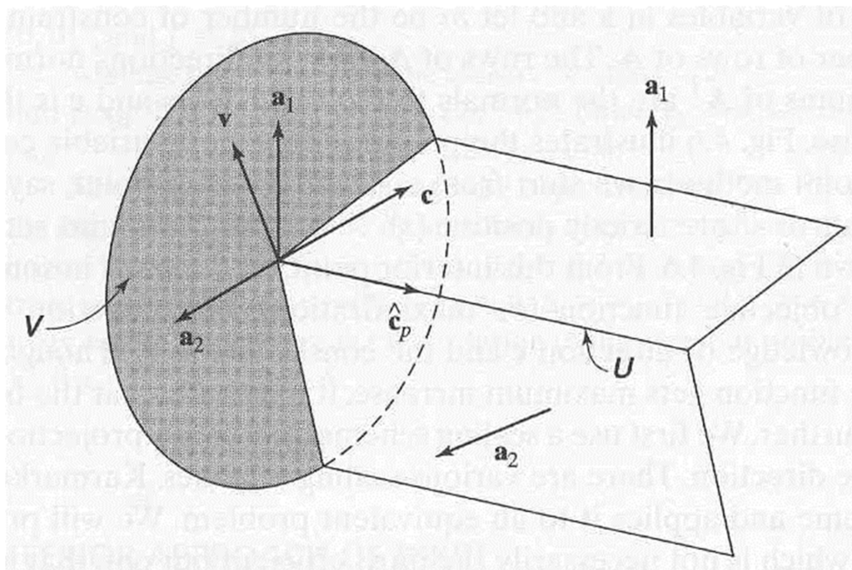
– Direction finding

$$\hat{\boldsymbol{A}}(\hat{\boldsymbol{x}}+\Delta\hat{\boldsymbol{x}})=\boldsymbol{b}\rightarrow\hat{\boldsymbol{A}}\Delta\hat{\boldsymbol{x}}=\boldsymbol{0}$$

$$\hat{\boldsymbol{c}}=\boldsymbol{v}+\hat{\boldsymbol{c}}_p=\hat{\boldsymbol{A}}^T\boldsymbol{y}+\hat{\boldsymbol{c}}_p\xrightarrow{\times\hat{A}}\hat{\boldsymbol{A}}\hat{\boldsymbol{c}}=\hat{\boldsymbol{A}}\left(\hat{\boldsymbol{A}}^T\boldsymbol{y}+\hat{\boldsymbol{c}}_p\right)\xrightarrow{\hat{A}\hat{c}_p=0}\hat{\boldsymbol{A}}\hat{\boldsymbol{A}}^T\boldsymbol{y}=\hat{\boldsymbol{A}}\hat{\boldsymbol{c}}$$

$$\hat{\boldsymbol{c}}_p=\hat{\boldsymbol{c}}-\hat{\boldsymbol{A}}^T\boldsymbol{y}=\left[\boldsymbol{I}-\hat{\boldsymbol{A}}^T\left(\hat{\boldsymbol{A}}\hat{\boldsymbol{A}}^T\right)^{-1}\hat{\boldsymbol{A}}\right]\hat{\boldsymbol{c}}=\boldsymbol{P}\hat{\boldsymbol{c}}$$

$$\boldsymbol{P}=\boldsymbol{I}-\hat{\boldsymbol{A}}^T\left(\hat{\boldsymbol{A}}\hat{\boldsymbol{A}}^T\right)^{-1}\hat{\boldsymbol{A}}$$



– Move distance

$$\Delta\hat{\boldsymbol{x}}=\alpha\sigma\hat{\boldsymbol{c}}_p\leftarrow\sigma=-\frac{1}{\hat{\boldsymbol{c}}_p{}^*}$$

$$\left(\hat{\boldsymbol{c}}_p{}^*:\text{most negative component of }\hat{\boldsymbol{c}}_p\right)$$

$$\Delta\boldsymbol{x}=\alpha\sigma\boldsymbol{D}\hat{\boldsymbol{c}}_p$$