4.80 (units: N, m)

$$\begin{cases} \underset{R,t}{\text{minimize}} \;\; f = 2\rho l \pi R t \rightarrow (246615.02)Rt \\ \text{subject to} \\ g_1 = \dfrac{P}{2\pi R t} \le \sigma_a \rightarrow g_1 = \dfrac{7957.75}{Rt} - 2.5\times 10^8 \le 0 \\ g_2 = P \le \dfrac{\pi^3 E R^3 t}{4 l^2} \rightarrow g_2 = \left(5.0\times 10^4\right) - \left(6.5113\times 10^{10}\right) R^3 t \le 0 \\ g_3 = \dfrac{R}{t} \le 50 \rightarrow g_3 = \dfrac{R}{t} - 50 \le 0 \\ g_4 = R \ge 0 \rightarrow g_4 = -R \le 0 \\ g_5 = t \ge 0 \rightarrow g_6 = -t \le 0 \end{cases}$$

$$L = (246615.02)Rt + u_1\left(\frac{7957.75}{Rt} - 2.5\times 10^8 + s_1^{\,2}\right) + u_2\left[\left(5.0\times 10^4\right) - \left(6.5113\times 10^{10}\right)R^3 t + s_2^{\,2}\right]$$

$$+ u_3\left(\frac{R}{t} - 50 + s_3^{\,2}\right) + u_4\left(-R + s_4^{\,2}\right) + u_5\left(-t + s_5^{\,2}\right)$$

$$\begin{cases} \dfrac{\partial L}{\partial R} = (246615.02)t - u_1\left(\dfrac{7957.75}{R^2 t}\right) - \left(1.9534\times 10^{11}\right)R^2 t u_2 + \dfrac{u_3}{t} - u_4 = 0 \\ \dfrac{\partial L}{\partial t} = (246615.02)R - u_1\left(\dfrac{7957.75}{R t^2}\right) - \left(6.5113\times 10^{10}\right)R^3 u_2 - \dfrac{R}{t^2} u_3 - u_5 = 0 \\ \dfrac{\partial L}{\partial s_i} = s_i u_i = 0 \quad (u_i \ge 0) \quad i = 1,\ldots,5 \end{cases}$$

$$\rightarrow 32 \text{ cases} \rightarrow u_1 = s_2 = s_3 = u_4 = u_5 = 0 \rightarrow u_2 = 3.056\times 10^{-4}, u_3 = 0.3038$$

$$R^* = 7.87\times 10^{-2} m,\; t^* = 1.57\times 10^{-3} m,\;\; f^* = 30.56 kg$$

4.81 (units: N, m)

$$\underset{R_o, R_i}{\text{minimize}} \quad f = \pi\rho l\left(R_o^{\ 2} - R_i^{\ 2}\right) \rightarrow \left(1.2331\times10^5\right)\left(R_o^{\ 2} - R_i^{\ 2}\right)$$

subject to

$$g_1 = \frac{P}{\pi\left(R_o^{\ 2} - R_i^{\ 2}\right)} \le \sigma_a \rightarrow g_1 = \frac{15915.5}{\left(R_o^{\ 2} - R_i^{\ 2}\right)} - 2.5\times10^8 \le 0$$

$$g_2 = P \le \frac{\pi^3 E\left(R_o^{\ 4} - R_i^{\ 4}\right)}{16l^2} \rightarrow g_2 = 50000 - .\left(1.62783\times10^{10}\right)\left(R_o^{\ 4} - R_i^{\ 4}\right) \le 0$$

$$g_3 = \frac{\left(R_o + R_i\right)}{2\left(R_o - R_i\right)} \le 50 \rightarrow g_3 = \frac{\left(R_o + R_i\right)}{2\left(R_o - R_i\right)} - 50 \le 0$$

$$g_4 = R_o \ge 0 \rightarrow g_4 = -R_o \le 0$$

$$g_5 = R_i \ge 0 \rightarrow g_5 = -R_i \le 0$$

$$L = \left(1.2331\times10^5\right)\left(R_o^{\ 2} - R_i^{\ 2}\right) + u_1\left[\frac{15915.5}{\left(R_o^{\ 2} - R_i^{\ 2}\right)} - 2.5\times10^8 + s_1^{\ 2}\right] + u_2\left[50000 - \left(1.62783\times10^{10}\right)\left(R_o^{\ 4} - R_i^{\ 4}\right) + s_2^{\ 2}\right]$$

$$+ u_3\left[\frac{\left(R_o + R_i\right)}{2\left(R_o - R_i\right)} - 50 + s_3^{\ 2}\right] + u_4\left(-R_o + s_4^{\ 2}\right) + u_5\left(-R_i + s_5^{\ 2}\right)$$

$$\begin{cases} \dfrac{\partial L}{\partial R_o} = \left(2.4662\times10^5\right)R_o - \dfrac{2(15915.5)R_o}{\left(R_o^{\ 2} - R_i^{\ 2}\right)^2}u_1 - 4\left(1.62783\times10^{10}\right)R_o^{\ 3}u_2 - \dfrac{R_i}{\left(R_o - R_i\right)^2}u_3 - u_4 = 0 \\ \dfrac{\partial L}{\partial R_i} = -\left(2.4662\times10^5\right)R_i + \dfrac{2(15915.5)R_i}{\left(R_o^{\ 2} - R_i^{\ 2}\right)^2}u_1 + 4\left(1.62783\times10^{10}\right)R_i^{\ 3}u_2 + \dfrac{R_o}{\left(R_o - R_i\right)^2}u_3 - u_5 = 0 \\ \dfrac{\partial L}{\partial s_i} = s_i u_i = 0 \quad \left(u_i \ge 0\right) \quad i = 1, \ldots, 5 \end{cases}$$

$$\rightarrow 32 \text{ cases} \rightarrow u_1 = s_2 = s_3 = u_4 = u_5 = 0 \rightarrow u_2 = 3.056\times10^{-4}, u_3 = 0.3055$$

$$R_o^{\ *} = 7.95\times10^{-2}m, \ R_o^{\ *} = 7.79\times10^{-2}m, \ f^{\ *} = 30.56kg$$

4.150

$$\nabla g_1 = \begin{bmatrix} -\dfrac{P_u}{\sqrt{2}x_1{}^2} - \dfrac{P_v}{\sqrt{2}\left(x_1+\sqrt{2}x_2\right)^2} \\ -\dfrac{P_v}{\left(x_1+\sqrt{2}x_2\right)^2} \end{bmatrix}, \quad \boldsymbol{H}g_1 = \begin{bmatrix} \dfrac{\sqrt{2}P_u}{x_1{}^3} + \dfrac{\sqrt{2}P_v}{\left(x_1+\sqrt{2}x_2\right)^3} & \dfrac{2P_v}{\left(x_1+\sqrt{2}x_2\right)^3} \\ \dfrac{2P_v}{\left(x_1+\sqrt{2}x_2\right)^3} & \dfrac{2\sqrt{2}P_v}{\left(x_1+\sqrt{2}x_2\right)^3} \end{bmatrix}$$

$$\nabla g_2 = \begin{bmatrix} -\dfrac{\sqrt{2}P_v}{\left(x_1+\sqrt{2}x_2\right)^2} \\ -\dfrac{2P_v}{\left(x_1+\sqrt{2}x_2\right)^2} \end{bmatrix}, \quad \boldsymbol{H}g_2 = \begin{bmatrix} \dfrac{2\sqrt{2}P_v}{\left(x_1+\sqrt{2}x_2\right)^3} & \dfrac{4P_v}{\left(x_1+\sqrt{2}x_2\right)^3} \\ \dfrac{4P_v}{\left(x_1+\sqrt{2}x_2\right)^3} & \dfrac{4\sqrt{2}P_v}{\left(x_1+\sqrt{2}x_2\right)^3} \end{bmatrix}$$

$$\left.\begin{array}{l} \left.\begin{array}{l} g_1 : \text{positive demidefinite} \rightarrow \text{convex} \\ g_2 : \text{positive demidefinite} \rightarrow \text{convex} \\ g_3, g_4 : \text{linear} \rightarrow \text{convex} \end{array}\right\} \rightarrow \text{constraint set is convex} \\ f : \text{linear} \rightarrow \text{convex} \end{array}\right\} \rightarrow \text{convex problem}$$

4.152

$$L = 2\sqrt{2}x_1 + x_2 + u_1\left\{\frac{1}{\sqrt{2}}\left[\frac{P_u}{x_1} + \frac{P_v}{x_1+\sqrt{2}x_2}\right] - 20000\right\} + u_2\left\{\frac{\sqrt{2}P_v}{x_1+\sqrt{2}x_2} - 20000\right\} + u_3\left(-x_1\right) + u_4\left(-x_2\right)$$

$g_1 = g_2 = 0 (= u_3 = u_4)$

$$\left.\begin{array}{l} \dfrac{\partial L}{\partial x_1} = 2\sqrt{2} + u_1\left[-\dfrac{P_u}{\sqrt{2}x_1{}^2} - \dfrac{P_v}{\sqrt{2}\left(x_1+\sqrt{2}x_2\right)^2}\right] + u_2\left[-\dfrac{\sqrt{2}P_v}{\left(x_1+\sqrt{2}x_2\right)^2}\right] = 0 \\ \dfrac{\partial L}{\partial x_2} = 1 + u_1\left[-\dfrac{P_v}{\left(x_1+\sqrt{2}x_2\right)^2}\right] + u_2\left[-\dfrac{2P_v}{\left(x_1+\sqrt{2}x_2\right)^2}\right] = 0 \end{array}\right\} \rightarrow u_2 \ge 0 \rightarrow P_v - 3P_u \ge 0 \rightarrow \tan\theta \ge 3$$

$$\left.\begin{array}{l} g_1 = \dfrac{1}{\sqrt{2}}\left[\dfrac{P_u}{x_1} + \dfrac{P_v}{x_1+\sqrt{2}x_2}\right] - 20000 = 0 \rightarrow x_1 = \dfrac{P_u}{10000\sqrt{2}} \\ g_2 = \dfrac{\sqrt{2}P_v}{x_1+\sqrt{2}x_2} - 20000 = 0 \rightarrow x_1 + \sqrt{2}x_2 = \dfrac{\sqrt{2}P_v}{20000} \end{array}\right\} \rightarrow x_2 = \frac{P_v - P_u}{20000} \ge 0 \rightarrow \tan\theta \ge 1$$

optimal solution only when $\theta \ge 71.57°$

5.50

(0) problem formulation

$$\text{Minimize} \quad f = 400\left(0.5\pi D^2 + \pi DH\right)$$

$$\text{subject to} \;\; h_1 = \frac{\pi D^2 H}{4} - 250\pi = 0$$

$$g_1 = H - 8D \le 0$$

(1) check for convexity

$$\nabla f = \begin{bmatrix} 400\pi D + 400\pi H \\ 400\pi D \end{bmatrix}, \quad H = \begin{bmatrix} 400\pi & 400\pi \\ 400\pi & 0 \end{bmatrix}$$

since Hessian of the cbjective function is NOT positive definite,
this is NOT a convex programming problem.

(2) KT necessary conditions

$$L = 400\left(0.5\pi D^2 + \pi DH\right) + v_1\left(\frac{\pi D^2 H}{4} - 250\pi\right) + u_1\left(H - 8D\right)$$

$$\frac{\partial L}{\partial D} = 400\pi D + \pi H + v_1\frac{\pi DH}{2} + u_1(-8) = 0$$

$$\frac{\partial L}{\partial H} = 400\pi D + v_1\frac{\pi D^2}{4} + u_1 = 0$$

$$h_1 = 0$$

$$g_1 \le 0,\; u_1 g_1 = 0,\; u_1 \ge 0$$

(3) solve the KT conditions

$i)\; g_1 = 0 \to H = 8D \to D = 5, H = 40, v_1 = -226.7, u_1 = -1832.5 < 0(\times)$

$ii)\; u_1 = 0 \to D = H = 10, v_1 = -160, g_1 = H - 8D = -70 < 0 \to f^* = 60000\pi$

sufficiency check:

$$\nabla^2 L = \nabla^2 f + v_1 \nabla^2 h_1 = \begin{bmatrix} 400\pi & 400\pi \\ 400\pi & 0 \end{bmatrix} - 160\begin{bmatrix} 0.5\pi H & 0.5\pi D \\ 0.5\pi D & 0 \end{bmatrix} = \begin{bmatrix} 400\pi - 80\pi H & 400\pi - 80\pi D \\ 400\pi - 80\pi D & 0 \end{bmatrix}$$

$$\nabla^2 L\left(\boldsymbol{x}^*\right) = \begin{bmatrix} -400\pi & -400\pi \\ -400\pi & 0 \end{bmatrix} : \text{NOT positive definite!}$$

$$\nabla h_1\left(\boldsymbol{x}^*\right)^T \boldsymbol{d} = 25\pi\begin{bmatrix} 2 & 1 \end{bmatrix}\begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = 0 \to \boldsymbol{d} = c\begin{bmatrix} 1 & -2 \end{bmatrix}^T$$

$$\boldsymbol{d}^T \nabla^2 L\left(\boldsymbol{x}^*\right)\boldsymbol{d} = -400\pi c^2\begin{bmatrix} 1 & -2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \end{bmatrix} = 1200\pi c^2 > 0 \to \text{isolated local minimum}$$

(4) post-optimality analysis

$h_1 : 250\pi \rightarrow 255\pi$

$\Delta f = -v_1 \Delta b = -(-160)(255\pi - 250\pi) = 800\pi$

4.1

1	2	3	4	5	6	7	8	9	10
T	F	T	F	T	F	F	F	T	F
11	12	13	14	15	16	17	18	19	20
T	T	F	F	T	F	F	T	T	T

4.21

1	2	3	4	5	6	7	8	9	10
T	T	T	F	T	F	F	F	F	F

4.43

1	2	3	4	5	6	7	8	9	10
F	T	F	T	F	T	T	T	F	T
11	12	13	14	15					
F	T	F	F	F					

4.132

1	2	3	4	5	6	7	8	9	10
T	T	T	F	T	F	F	F	F	T