Final Exam

06/17/2009

1. This problem is about the equation

$$-u''(x)=1,\quad u(0)=u(1)=0$$

   (1) Find the exact solution. (10 pts)
   (2) Let h = 1/3, and use the piecewise linear hat functions ϕ_1 and ϕ_2, and the test functions $V_1=\phi_1$ and $V_2=\phi_2$ to calculate the 2 x 2 finite element matrix K. Calculate the vector F. (10 pts)
   (3) Find the coefficient vector U satisfying $KU=F$. Sketch the graph of $U(x)=U_1\phi_1(x)+U_2\phi_2(x)$ and the graph of the solution $u(x)$. What is the maximum error $|U(x)-u(x)|$? (10 pts):

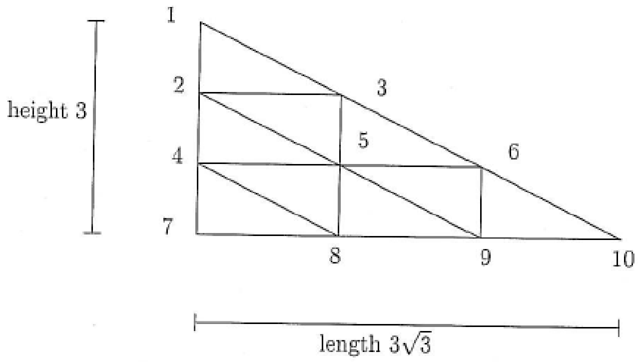
2. This is about the velocity field $v(x,y)=(0,x)=w(x,y)$.
   (1) Check that div w = 0 and find a stream function $s(x,y)$. Draw the streamlines in the xy plane and show some velocity vectors. (10 pts)
   (2) Is this shear flow a gradient a gradient field (v = grad u) or is there rotation? If you believe u exists, find it. If you believe there is rotation, explain how this is possible with the streamlines you drew in (1). (10 pts)

3. This problem concerns the Finite Element Method for the triangular domain. This is a right triangle, with bottom edge of length $3\sqrt{3}$ and left edge of length 3. Number the nodes as in the figure. We consider the following BVP:

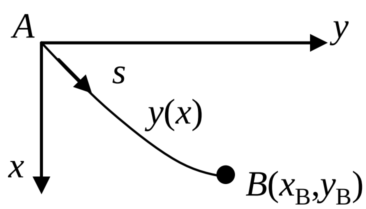
$$-\Delta u=1 \text{ in } \Omega,\quad u|_{\partial\Omega}=0$$

   (1) Each small triangle in the mesh is the same size. Compute the 3 x 3 element matrix Ke for this problem for an individual triangle. (10 pts)
   (2) Compute the 5th row of the full element matrix K. (10 pts)
   (3) Use the boundary conditions and the load function to get a reduced, non-singular matrix Kr and load vector Fr and solve for the interior values of U where U solves $KU=F$ with appropriate boundary conditions as usual. (10 pts)

4. [Brachistochrone Problem] In June 1696, Johann Bernoulli set the following problem before the scholars of his time. "Given two points A and B in a vertical plane, find the path from A to B along which a particle of mass m will slide under the force of gravity, without friction, in the shortest time". The term "brachistochrone" derives from the Greek brachistos (shortest) and chronos (time).
   (1) Drive the integral to be stationary as $t=\int_0^{x_B}\left[\frac{1+(y')^2}{2gx}\right]^{1/2}dx$ since potential energy is converted to kinetic energy as the particle moves down the path. (10 pts)
   (2) Find the path. (10 pts)



Problem 3



Problem 4