Final Exam

06/16/2010

1. Suppose to use linear finite elements (hat functions $\phi(x)$ = trial functions $V(x)$). The equation has $c(x)=1+x$ and a point load:

$$-\frac{d}{dx}\left((1+x)\frac{du}{dx}\right)=\delta\left(x-\frac{1}{2}\right) \text{ with } u(0)=u'(1)=0 \text{ [fixed-free]}$$

Take $h=1/3$ with two hats and a half-hat as in the notes.

(1) On the middle interval from 1/3 to 2/3, $U(x)$ goes linearly from U_1 to U_2.

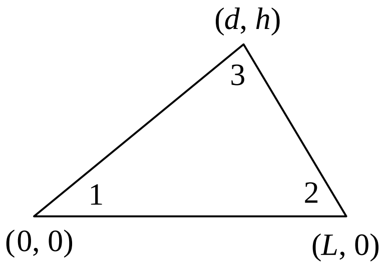
Compute $\int_{1/3}^{2/3} c(x)(U'(x))^2 dx = [U_1 \quad U_2]\begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix}\begin{bmatrix} U_1 \\ U_2 \end{bmatrix}$ and $\int_{1/3}^{2/3} \delta\left(x-\frac{1}{2}\right)U(x)dx = [U_1 \quad U_2]\begin{bmatrix} ? \\ ? \end{bmatrix}$. (5 pts)

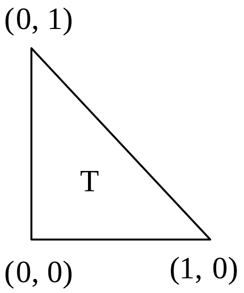
(2) On the first and third intervals, compute similar integrations

$[U_1][?][U_1]$ and $[U_1][?]$; $[U_2 \quad U_3]\begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix}\begin{bmatrix} U_2 \\ U_3 \end{bmatrix}$ and $[U_2 \quad U_3]\begin{bmatrix} ? \\ ? \end{bmatrix}$ (10 pts)

(3) What would be the overall finite equation $KU=F$? (not to solve) (5 pts)

2. If $U=a+bx+cy$ on the triangle below, find *b* and *c* from the equations $U=U_1$, $U=U_2$, $U=U_3$ at the nodes.





(1) Show that the gradient matrix G is $\begin{bmatrix} b \\ c \end{bmatrix} = GU = \begin{bmatrix} -\frac{1}{L} & \frac{1}{L} & 0 \\ \frac{d}{Lh}-\frac{1}{h} & -\frac{d}{Lh} & \frac{1}{h} \end{bmatrix}\begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix}$. (10 pts)

(2) For the previous matrix multiply G^TG by the area $Lh/2$ to find the element stiffness matrix K_e. For the standard triangle T show that $K_e = KT = \frac{1}{2}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$. (10 pts)

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3. Which of these 5 equations can be solved? If the equation has a solution, please find one. If not why not? (20 pts)

   (1) div $\begin{bmatrix} \partial u/\partial x \\ \partial u/\partial y \end{bmatrix} = 1$

   (2) div $\begin{bmatrix} \partial s/\partial y \\ -\partial s/\partial x \end{bmatrix} = 1$

   (3) $u_{xx} + u_{yy} = 0$ in the unit circle and $u(1,\theta) = \sin 4\theta$ around the boundary

   (4) Find a family of curves $u(x, y) = C$ that is everywhere perpendicular to the family of curves $x + x^2 - y^2 = C$

   (5) $d^4u/dx^4 = \delta(x)$ [point load at $x = 0$, not requiring boundary conditions, any solution $u(x)$ is OK]

4. Minimize $\|Au - b\|^2$ with $Bu = d$, in three ways: $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, $b = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 3 \end{bmatrix}$, $d = 20$

   (1) (Large penalty) Minimize $\|Au - b\|^2 + \alpha \|Bu - d\|^2$ (6 pts)

   (2) (Lagrange multiplier) $L = \frac{1}{2}\|Au - b\|^2 + w(Bu - d)$ (6 pts)

   (3) (Nullspace method) Find the complete solution $u = u_r + u_n$ to $Bu - d$. Choose z to minimize $\|Au - b\|^2$. (8 pts)

5. [Isoperimetric Problem] Find that curve C having given length l which encloses a maximum area. The area bounded by C is $A = \frac{1}{2}\int_C (xdy - ydx)$ while the arc length is $l = \frac{1}{2}\int_C \sqrt{1 + y'^2}\, dx$. (20 pts)