Midterm Exam

04/18/2012

1. This problem concerns solving equations with point masses (delta functions).

(1) Solve the following equation for $u(x)$ with a point mass at $x = a$: (10 pts)

$$\begin{cases} -\dfrac{d^2}{dx^2}u = \delta(x-a) \\ u(0) = 1,\ u'(1) = -1 \end{cases}$$

(2) Use part (1) to solve the following equation for $U(x)$: (5 pts)

$$\begin{cases} -\dfrac{d^2}{dx^2}U = x^2 \\ U(0) = 1,\ U'(1) = -1 \end{cases}$$

(Hint: If $G(x,a)$ is the solution to part (1), then $U(x) = \int_0^1 G(x,a)a^2 da + \text{null solution}$. Don't forget to check the boundary conditions and add an appropriate nullspace solution.)

(3) Solve the following matrix equation for $a = 1$ and also for $a = 2$: (10 pts)

$$K_3 u = \delta_a \quad \text{where as usual } K_3 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \text{ and } \delta_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},\ \delta_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

2. Suppose u_1, u_2, u_3, u_4 are unknowns at meshpoints $x = \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{4}{4}$ with $h = \frac{1}{4}$ and boundary value $u_0 = 0$ at one end.

(1) Write down a 4 by 4 backward difference matrix A, which acts on u to produce differences $e_i = \dfrac{u_i - u_{i-1}}{h}$. (5 pts)

(2) Find the symmetric part $S = \frac{1}{2}\left(A + A^T\right)$ and the antisymmetric part $AS = \frac{1}{2}\left(A - A^T\right)$. What words would you use for this particular S and AS? (5 pts)

(3) Compute $K = A^T C A$ when $C = \text{diag}(c_1, c_2, c_3, c_4)$ is a diagonal matrix with positive c's. Show by some test that K is positive definite. (Explain your test exactly.) (5 pts)

(4) If $f = (0,0,0,1)$ is a unit force pulling the 4th mass (at $x = 1$) downward, what will be the displacement u_4 at the bottom of the line of springs? What entry of K^{-1} does this tell you? Use physical reasoning with 4 springs, much quicker than solving 4 equations. (10 pts)

3.

(1) Find the eigenvalues $\lambda_1, \lambda_2, \lambda_3$ and unit eigenvectors y_1, y_2, y_3 of B. (10 pts)

(Hint: one eigenvector is $(1,0,-1)/\sqrt{2}$)

$$B = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

(2) Factor B into $Q\Lambda Q^T$ with $Q^{-1} = Q^T$. Draw a graph of the energy function $f(u_1, u_2, u_3) = \frac{1}{2} u^T Bu$. This is a surface in 4-dimensional u_1, u_2, u_3, f space so your graph may not be perfect. Describe it in 1 sentence. (10 pts)

(3) What differential equation with what boundary conditions on $y(x)$ at $x = 0$ and 1 is the continuous analogy of $By = \lambda y$? What are the eigenfunctions $y(x)$ and eigenvalues λ in this differential equation? At which $x's$ would you sample the first three eigenfunctions to get the three eigenvectors in part (1) (10 pts)

4. Consider fitting the data $(-1, f_1), (0, f_2), (1, f_3)$ to the curve $y = C + Dt^2$.

(1) What matrix equation $Au = f$ would we like to solve (but probably cannot) for the best choice of C and D? Is $A^T A$ positive definite or only positive sei-definite? (5 pts)

(2) Decide whether $(A^T A)^{100}$ is very large, nearly 0, or something else? (Explain) (5 pts)

(3) Now consider fitting the same three points with the curve $y = C + Dt^2 + E(e^t + e^{-t} - 2)$. What is the new A? Are the columns of this new matrix independent? What does that mean? (Explain your answer.) (10 pts)