Final Exam

06/16/2014

1.

(1) Suppose $w(x,y)=(w_1(x,y),0)$ is a flow field. With $w_2=0$ write down the remaining (not zero) terms in Green's formula for the integral $\iint(\text{grad}\,u)\cdot w dxdy$ in the unit square $0\le x\le 1,\ 0\le y\le 1$. Substitute for n and ds when you know what they are for this square. (10 pts)

(2) A one-dimensional formula on any horizontal line $y=y_0$ is integration by parts:

$$\int_{x=0}^{1}\frac{du}{dx}w_1(x)dx=-\int_{x=0}^{1}u(x)\frac{dw_1}{dx}dx+uw_1(x=1)-uw_1(x=0)$$

Here u and w_1 are $u(x,y_0)$ and $w_1(x,y_0)$ since $y=y_0$ is fixed. How do you derive your Green's formula in part (1) from this one-dimensional formula? (5 pts)

(3) Find all vector fields of this form $(w_1(x,y),0)$ that can be velocity that can be velocity fields $v=w=(w_1(x,y),0)$ in potential flow [so $v=\text{grad}\,u$ and $\text{div}\,w=0$ as usual]. (10 pts)

2.

(1) Find a 4th degree polynomial $s(x,y)$ with only 2 terms that solves Laplace's equation. Please draw a box around your answer $s(x,y)$. (10 pts)

(2) In the xy plane draw all the solutions to $s(x,y)=0$. Then in the same picture roughly draw the curve $s(x,y)=c$ that goes through the particular point $(x,y)=(2,1)$. (10 pts)

(3) If the curves $s(x,y)=c$ are the streamlines of a potential flow (in the usual framework), what is the corresponding velocity $v(x,y)=w(x,y)$?. (10 pts)

3. Consider the boundary value problem with Dirichlet boundary conditions:

$$-\frac{d}{dx}\left(\rho(x)\frac{du}{dx}\right)=1 \text{ with } u(0)=0 \text{ and } u(1)=0 \text{ where } \rho(x)=\begin{cases}1 & \text{if } x\le 1/2\\ 2 & \text{if } x\ge 1/2\end{cases}$$

(1) Write out the weak form of the boundary value problem. (10 pts)

(2) Let $N=3,\ \Delta x=\frac{1}{4},\ x_i=i\Delta x$. For $i=1,\ 2,\ 3$, let $\phi_i(x)$ be the piecewise linear functions that satisfy $\phi_i(x_j)=\begin{cases}1 & \text{if } i=j\\ 0 & \text{if } i\ne j\end{cases}$. Write out the linear system that needs to be solved under a finite element method with basis functions $\phi_1,\ \phi_2,\ \phi_3$. (20 pts)

4. The shortest curve connecting two points is a straight line. Suppose we cannot go in a straight line because of a constraint. When the constraint is $\int u(x)dx = A$, find the shortest curve $u(x)$ between $u(0)=a$ and $u(1)=b$ that has area A below it. (20 pts)

5. Minimize $\|Au-b\|^2$ with $Bu=d$, in three ways: $A=\begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix}$, $b=\begin{bmatrix}0\\ 0\end{bmatrix}$, $B=\begin{bmatrix}1 & 3\end{bmatrix}$, $d=20$

   (1) (Large penalty) Minimize $\|Au-b\|^2+\alpha\|Bu-d\|^2$ (5 pts)

   (2) (Lagrange multiplier) $L=\frac{1}{2}\|Au-b\|^2+w(Bu-d)$ (5 pts)

   (3) (Nullspace method) Find the complete solution $u=u_r+u_n$ to $Bu-d$. Choose z to minimize $\|Au-b\|^2$. (5 pts)