Midterm Exam

04/21/2014

1. Suppose I know the boundary conditions $u(0)=0$ and $u(1)=0$. The stepsize $h=\frac{1}{4}$ divides the interval [0, 1] into four equal parts. Three unknown values $u_1,\ u_2,\ u_3$ go into the vector u.

(1) What 3 by 3 matrix M would use second differences for $-u''$ and centered first differences for u', so that Mu approximates $-u''+u'=-\frac{d^2u}{dx^2}+\frac{du}{dx}$ at the meshpoints $x=\frac{1}{4},\ \frac{2}{4},\ \frac{3}{4}$? (10 pts)

(2) The exact solution $u(x)$ to $-u''+u'=\delta(x-a)$ would have a two-part formula: $u=\begin{cases} A+Be^x & \text{for } x\le a \\ C+De^x & \text{for } x\ge a \end{cases}$

Why did I write $A+Be^x$? What four equations (using the conditions at $x=0$ and $x=1$ and $x=a$) determine the four constants $A,\ B,\ C,\ D$? Not necessary to solve these equations. (10 pts)

2. Draw a line of 3 springs and 2 masses with both ends fixed: $u_0=u_3=0$. The spring constants are $c_1=2,\ c_2=1,\ c_3=2$.

(1) Find the stiffness matrix $K=A^TCA$. (7 pts)

(2) One eigenvector of K is $(1,\ 1)^T$. Find its eigenvalue. (5 pts)

(3) Write down nonzero solution to the equation $\frac{d^2u}{dt^2}+Ku=0$. (8 pts)

3. We are given observations $b_1,\ b_2,\ b_3$ at times $t=-1,\ 0,\ 1$.

(1) Find the closest straight line $C+Dt$ to those three observations. First tell me which three equations $Au=b$ would be solved for $u=\begin{bmatrix} C \\ D \end{bmatrix}$ if the three points $(-1,b_1)$ and $(0,b_2)$ and $(1,b_3)$ were exactly on a line.

Then find the least squares solution $\hat{u}=\begin{bmatrix} \hat{C} \\ \hat{D} \end{bmatrix}$. (8 pts)

(2) For which $b_1,\ b_2,\ b_3$ does the line go exactly through the three points so that $Au=b$ is solved? (5 pts)

(3) Find an example of $b_1,\ b_2,\ b_3$ so that the best line is just the zero line with $\hat{C}=\hat{D}=0$. Check that this vector $b_1,\ b_2,\ b_3$ is orthogonal to both columns of A. (7 pts)

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4. Suppose a graph has $m=3$ edges, going from node 1 at the center to three outer points (nodes 2, 3, 4)

(1) What is the incidence matrix A? What is its rank? (5 pts)

(2) What is the graph Laplacian $K=A^T A$? Is it positive definite or positive semidefinite and how do you know? (5 pts)

(3) What are all the solutions to Kirchhoff's Current Law $A^T w=0$? Add an edge 4 from node 2 to node 3 and find one more solution w. (10 pts)

5.

(1) A truss is in the shape of an X with 45° angles and 2 fixed nodes at the bottom. Based on the number *m* of bars and *n* of unknown displacements, how many independent solutions do you expect to $Au=0$? (5 pts)

(2) You can answer without writing down the matrix A. Given the components $u_1^H, u_1^V, \ldots, u_3^V$ for a full set of independent solutions of $Au=0$. Draw these mechanisms. (8 pts)

(3) What is row 1 of the matrix A (corresponding to upper left bar)? (7 pts)