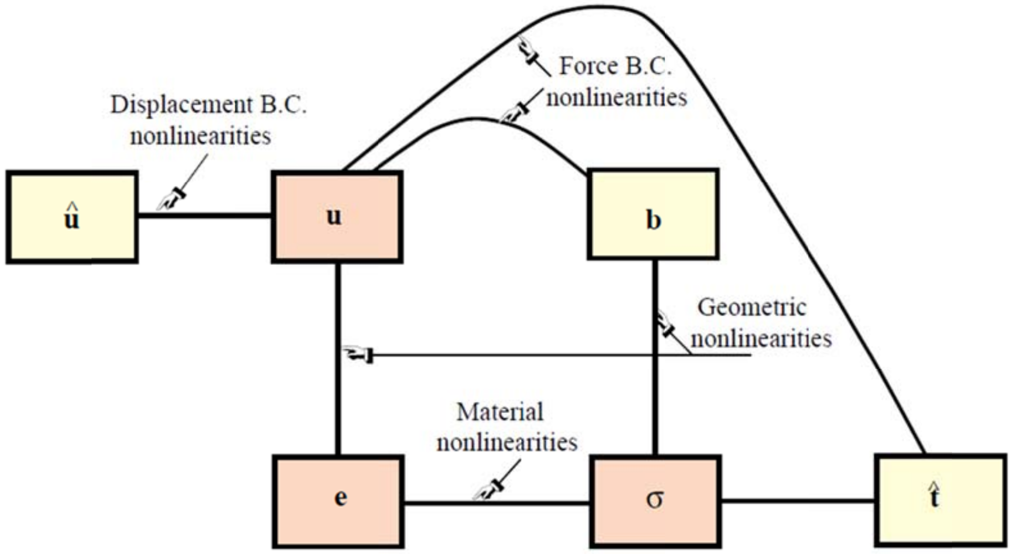
1. Tonti diagram (4 pts) nonlinearity relationship (6 pts), classification (2 pts) example (2 pts each)



- Geometric nonlinearity: Large strain (over 5%), Small strains but finite displacements and/or rotations, Linearized prebucking
- Material Nonlinearity: Structures undergoing nonlinear elasticity, plasticity, viscoelasticity, creep, or inelastic rate effects
- Force B.C. Nonlinearity: pressure loads of fluids, gyroscopic and non-conservative follower forces
- Displacement B.C. Nonlinearity: contact problem, contact problem

2. Critical Points: Response point at which K becomes singular (structural instability) (10 pts)

Limit point: path continues with no branching, but tangent is normal to λ(control parameter) axis

Bifurcation point: two or more paths cross and there is no unique tangent

$$\text{Let } \mathbf{z} \text{ such that } \mathbf{Kz}=0 \xrightarrow{\mathbf{K}\text{: symmetric}} \mathbf{z}^T\mathbf{K}=0$$

$$\mathbf{K}\dot{\mathbf{u}}=\mathbf{q}\dot{\lambda} \xrightarrow{dt} \mathbf{K}d\mathbf{u}=\mathbf{q}d\lambda \xrightarrow{\mathbf{z}^T} \mathbf{z}^T\mathbf{K}d\mathbf{u}=\mathbf{z}^T\mathbf{q}d\lambda=0$$

$$\rightarrow \begin{cases} \mathbf{z}^T\mathbf{q}\neq 0 \rightarrow d\lambda=0 \rightarrow \text{limit point} \begin{cases}\text{isolated}\\ \text{multiple}\end{cases} \text{(10 pts)} \\ \mathbf{z}^T\mathbf{q}=0 \rightarrow \text{bifurcation or branching point} \begin{cases}\text{isolated}\\ \text{multiple}\end{cases} \text{(10 pts)} \\ \text{abrupt transition from one deformation mode to another mode} \end{cases}$$

3. Seth-Hill (SH) Family: Define a set of finite strain measures that depend on one parameter

suppose that the axial stretch is $\lambda \rightarrow$ define the axial strain as $e^{(m)} = \dfrac{\lambda^m - 1}{m}$

m: real number, which is usually chosen to be an integer in the range $[-2, 2]$

Name (m=index in SH family)	1D finite strain measure	Taylor series expansion about g=0	Comments
Green (m = 2)	$e^G = \frac{1}{2}(\lambda^2 - 1) = g + \frac{1}{2}g^2$ $= \frac{1}{2}(1-\hat{\lambda}^2)/\hat{\lambda}^2 = (1-\hat{g}^2/2)/(1-\hat{g})$	$g + g^2/2$	Widely used since the start of geometrically nonlinear FEM
Biot (m = 1)	$e^B = \lambda - 1 = g$ $= 1/\hat{\lambda} - 1 = \hat{g}/(1-\hat{g})$	g	Generalization of "engineering strain" to finite deformations. Becoming increasingly popular
Hencky (m = 0)	$e^H = \log\lambda = \log(1+g)$ $= -\log\hat{\lambda} = \log(1/(1-\hat{g}))$	$g - g^2/2 + g^3/3 - \ldots$	Used in finite elastoplasticity. Sometimes replaced by log-less approximations (see below)
Swainger ($m = -1$)	$e^S = 1 - 1/\lambda = g/(1+g)$ $= 1 - \hat{\lambda} = \hat{g}$	$g - g^2 + g^3 - \ldots$	Counterpart of Biot measure
Almansi (m=−2)	$e^A = \frac{1}{2}(1 - 1/\lambda) = (g + \frac{1}{2}g^2)/(1+g)$ $= \frac{1}{2}(1-\hat{\lambda}) = \frac{1}{2}(\hat{g} - \frac{1}{2}\hat{g}^2)/(1-\hat{g})$	$g - 3g^2/2 + 2g^3 - \ldots$	Counterpart of Green measure
Seth-Hill family (arbitrary m)	$e^{(m)} = \frac{1}{m}(\lambda^m - 1) = \frac{1}{m}((1+g)^m - 1)$	$g - (m-1)\,g^2/2$ $+ (m-1)(m-2)\,g^3/6 - \ldots$	Includes above five measures for m = 2, 1, 0, −1, −2, resp.
Midpoint	$e^M = 2(\lambda-1)/(\lambda+1) = g/(1+g/2)$ $= 2(1-\hat{\lambda})/(\hat{\lambda}+1) = \hat{g}/(1-\hat{g}/2)$	$g - g^2/2 + g^3/4 - \ldots$	(1,1) Pade approximant to Hencky strain that avoid logs. Not part of the Seth-Hill family
Bazant	$e^Z = \frac{1}{2}(e^B + e^S) = (g + g^2/2)/(1+g)$	$g - g^2/2 + g^3/2 - \ldots$	Another log-free approximant to Hencky strain; accuracy similar to e^M. Not part of the Seth-Hill family

Name (coeff in Seth-Hill family)	Finite strain tensor (a member of Seth-Hill family)	Conjugate stress tensor	Comments
Green-Lagrange (m = 2)	$\underline{e} = \frac{1}{2}(\mathbf{U}^2 - \mathbf{I}) = \frac{1}{2}(\mathbf{C}_R - \mathbf{I})$	$\underline{s} = J\,\mathbf{F}^{-1}\,\underline{\sigma}\,\mathbf{F}^{-T}$ $\underline{\sigma} = J^{-1}\mathbf{F}\,\underline{s}\,\mathbf{F}^T$	Second Piola-Kirchhoff tensor. Abbreviated to PK2 in text
Biot (m = 1)	$\underline{e}^B = \mathbf{U} - \mathbf{I} = \mathbf{C}_R^{1/2} - \mathbf{I}$	$\underline{s}^B = \frac{1}{2}(\underline{s}\,\mathbf{U} + \mathbf{U}\,\underline{s})$ $= \frac{1}{2}J(\mathbf{U}^{-1}\underline{\sigma}^R + \underline{\sigma}^R\mathbf{U}^{-1})$	Biot stress. In 2nd form, $\underline{\sigma}^R = \mathbf{R}^T\underline{\sigma}\,\mathbf{R}$ is back-rotated Cauchy stress
Hencky (m = 0)	$\mathbf{e}^H = \log(\mathbf{U}) = \frac{1}{2}\log(\mathbf{C}_R)$	Omitted (see text)	For isotropic material, $\underline{\sigma}^R$. Extremely complicated for anisotropic material
Swainger ($m = -1$)	$\mathbf{e}^S = \mathbf{I} - \mathbf{U}^{-1} = \mathbf{I} - \mathbf{C}_R^{-1/2}$	$\underline{s}^S = \frac{1}{2}(\underline{s}^A\mathbf{U}^{-1} + \mathbf{U}^{-1}\underline{s}^A)$ $= \frac{1}{2}J(\mathbf{U}\,\underline{\sigma}^R + \underline{\sigma}^R\,\mathbf{U})$	Swainger stress, rarely used. 2nd form in terms of back-rotated Cauchy stress
Almansi-Hamel (m=−2)	$\mathbf{e}^A = \frac{1}{2}(\mathbf{I} - \mathbf{U}^{-2}) = \frac{1}{2}(\mathbf{I} - \mathbf{C}_R^{-1})$	$\underline{s}^A = J\,\mathbf{F}^T\underline{\sigma}\,\mathbf{F}$ $\underline{\sigma} = J^{-1}\mathbf{F}^{-T}\underline{s}^A\mathbf{F}^{-T}$	Almansi stress

The Biot stress is sometimes called the Jaumann stress. The names "Hencky stress", "Swainger stress" and "Almansi stress" are introduced by pairing with the corresponding finite strain measures. The name "von Mises" sometimes appears along with Hencky.
If the material is isotropic, stress tensors referred to the reference configuration are coaxial with **U**, whereas those referred to the current configuration (e.g., the Cauchy stress tensor) are coaxial with **V**.

$$(\text{PK2} \rightarrow \text{Cauchy})\, \underline{\mathbf{s}} = J\mathbf{F}^{-1}\underline{\sigma}\mathbf{F}^{-T} \rightarrow \underline{\sigma} = J^{-1}\mathbf{F}\,\underline{\mathbf{s}}\,\mathbf{F}^{T}$$

$$(\text{Almansi-Hamel} \rightarrow \text{Cauchy})\, \underline{\mathbf{s}}^{A} = J\mathbf{F}^{T}\underline{\sigma}\mathbf{F} \rightarrow \underline{\sigma} = J^{-1}\mathbf{F}^{-1}\underline{\mathbf{s}}^{A}\mathbf{F}^{-T}$$

suppose $\mathbf{U} = \mathbf{diag}[\lambda_1, \lambda_2, \lambda_3], \underline{\mathbf{s}}^{B} = \mathbf{diag}\left[s_1{}^{B}, s_2{}^{B}, s_3{}^{B}\right], \underline{\sigma} = \mathbf{diag}[\sigma_1, \sigma_2, \sigma_3], \mathbf{R} = \mathbf{I}, J = \lambda_1\lambda_2\lambda_3$

$$(\text{Biot} \rightarrow \text{Cauchy})\, \underline{\sigma}^{R} = \mathbf{R}^{T}\underline{\sigma}\mathbf{R} = \underline{\sigma} \rightarrow \underline{\mathbf{s}}^{B} = \frac{1}{2}J\left(\mathbf{U}^{-1}\underline{\sigma}^{R} + \underline{\sigma}^{R}\mathbf{U}^{-1}\right) = J\mathbf{U}^{-1}\underline{\sigma}$$

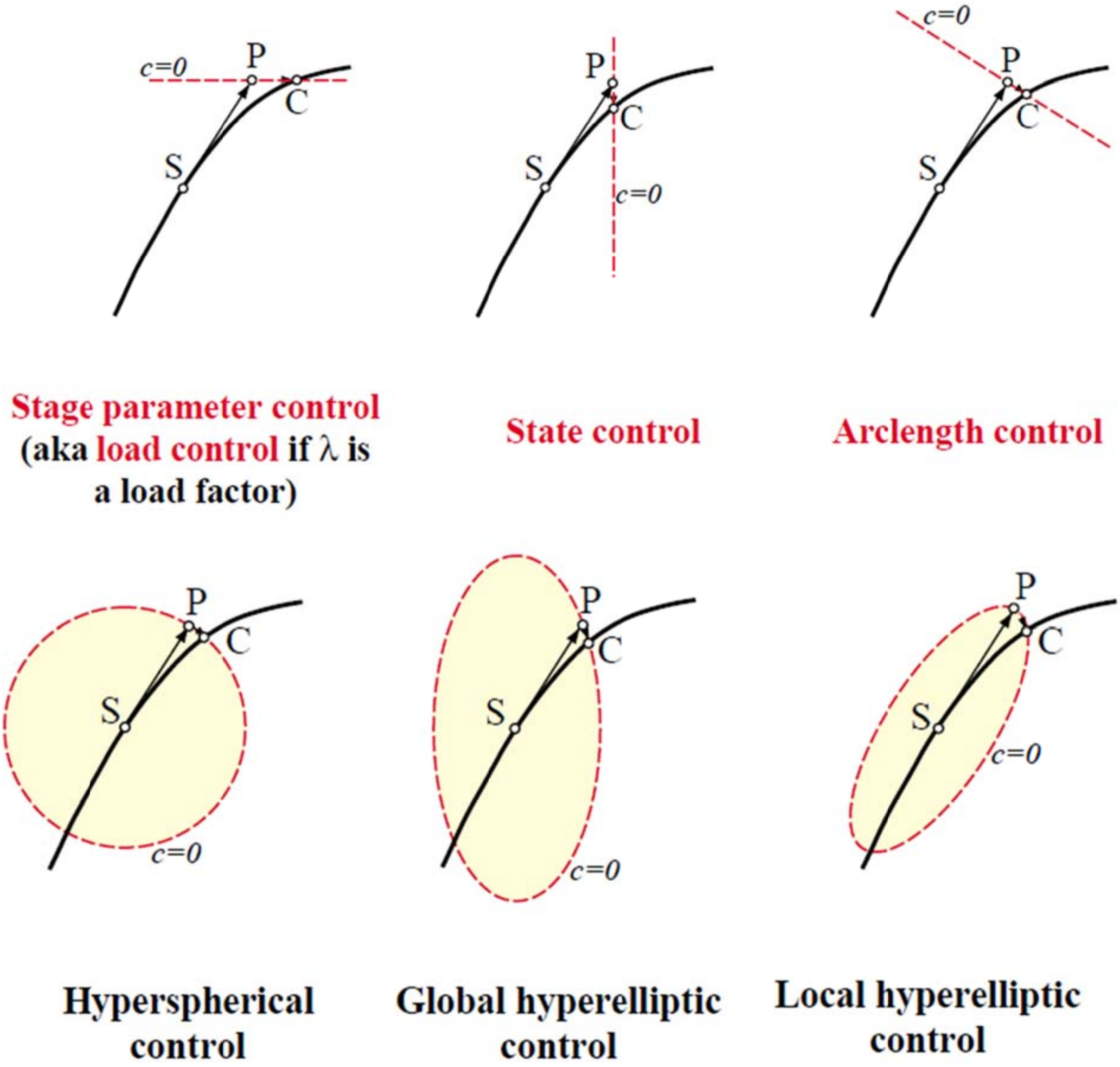
$$\rightarrow \underline{\sigma} = J^{-1}\mathbf{diag}\left[\lambda_1 s_1{}^{B}, \lambda_2 s_2{}^{B}, \lambda_3 s_3{}^{B}\right]$$

no need to solve Lyapunov-type equation

$$(\text{Green} \rightarrow \text{Cauchy})\, \mathbf{U} = \mathbf{F} = \mathbf{F}^{T}, \underline{\sigma} = J^{-1}\mathbf{F}\,\underline{\mathbf{s}}\,\mathbf{F}^{T} = J^{-1}\mathbf{U}\underline{\mathbf{s}}\mathbf{U} = J^{-1}\mathbf{diag}\left[\lambda_1{}^2 s_1, \lambda_2{}^2 s_2, \lambda_3{}^2 s_3\right]$$

general stress measure $\sigma^{(m)}$ conjugate to the SH strain $e^{(m)}$: $\sigma_i = J^{-1}\lambda_i{}^{m}s^{(m)}$

4.



strategy	constraint equation	rate form
load control	$c(\lambda) = \Delta\lambda_n - l_n = 0$	$\dot{\lambda} = 0$
state control	$c(\Delta\mathbf{u}_n) = \Delta\mathbf{u}_n{}^T\Delta\mathbf{u}_n - l_n{}^2 u_{ref}{}^2 = 0$	$2\Delta\mathbf{u}_n{}^T\dot{\mathbf{u}} = 0$
(displacement control)	$\rightarrow \Delta\tilde{\mathbf{u}}_n{}^T\Delta\tilde{\mathbf{u}}_n - l_n{}^2 = 0$	
modified state control	$c(\Delta\mathbf{u}_n) = \sqrt{\Delta\mathbf{u}_n{}^T\Delta\mathbf{u}_n} - l_n u_{ref} = 0$ $\lVert\Delta\mathbf{u}_n\rVert_2 - l_n u_{ref} = 0$	$\dfrac{\Delta\mathbf{u}_n}{\lVert\Delta\mathbf{u}_n\rVert_2}\dot{\mathbf{u}} = 0$
arclength control	$c(\Delta\mathbf{u}_n, \Delta\lambda_n) = \Delta s_n - l_n = \frac{1}{f_n}\left\lvert\mathbf{v}_n{}^T\Delta\mathbf{u}_n + \Delta\lambda_n\right\rvert - l_n = 0$	$\frac{1}{f_n}\left(\mathbf{v}_n\dot{\mathbf{u}} + \dot{\lambda}\right) = 0$
	$\rightarrow \Delta\tilde{s}_n - l_n = \frac{1}{\tilde{f}_n}\left\lvert\tilde{\mathbf{v}}_n{}^T\Delta\tilde{\mathbf{u}}_n + \Delta\lambda_n\right\rvert - l_n = 0$	$\frac{1}{\tilde{f}_n}\left(\mathbf{v}_n\mathbf{S}^2\dot{\mathbf{u}} + \dot{\lambda}\right) = 0$
global hyperelliptic control	$a_n{}^2\Delta\mathbf{u}_n{}^T\Delta\mathbf{u}_n + b_n{}^2(\Delta\lambda_n)^2 - l_n{}^2 = 0$	$2a^2\Delta\mathbf{u}\dot{\mathbf{u}} + 2b^2\Delta\lambda\dot{\lambda} = 0$
local hyperelliptic control	$a_n{}^2(\bar{\mathbf{u}} - \bar{\mathbf{u}}_n)^T\mathbf{S}(\bar{\mathbf{u}} - \bar{\mathbf{u}}_n) + b_n{}^2(\bar{\lambda} - \bar{\lambda}_n)^2 - l_n{}^2 = 0$	