Final Exam

12/19/2017

1. (30 pts) Figure shows a pin-jointed plane stress discretized with 2 elements and 3 nodes. Node 3 is fixed whereas 1 and 2 move over rollers as shown. The only nonzero applied load acts upward on node 1. Solve this problem by the Direct Stiffness Method. Start from the element stiffness equations, already incorporated the $E^e A^e / L^e$ factor below. The element stiffness equations in global coordinates are

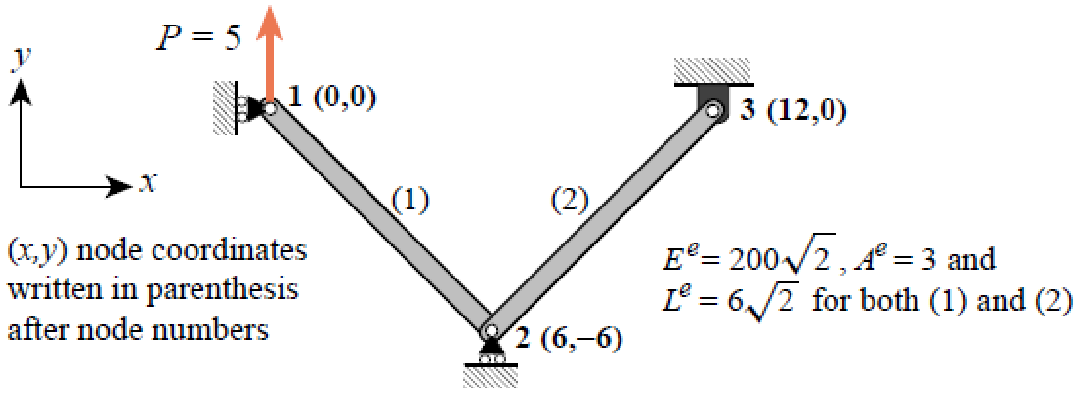
$$\begin{bmatrix} 50 & -50 & -50 & 50 \\ -50 & 50 & 50 & -50 \\ -50 & 50 & 50 & -50 \\ 50 & -50 & -50 & 50 \end{bmatrix} \begin{bmatrix} u_{x1}^{(1)} \\ u_{y1}^{(1)} \\ u_{x2}^{(1)} \\ u_{y2}^{(1)} \end{bmatrix} = \begin{bmatrix} f_{x1}^{(1)} \\ f_{y1}^{(1)} \\ f_{x2}^{(1)} \\ f_{y2}^{(1)} \end{bmatrix}, \quad \begin{bmatrix} 50 & 50 & -50 & -50 \\ 50 & 50 & -50 & -50 \\ -50 & -50 & 50 & 50 \\ -50 & -50 & 50 & 50 \end{bmatrix} \begin{bmatrix} u_{x2}^{(2)} \\ u_{y2}^{(2)} \\ u_{x3}^{(2)} \\ u_{y3}^{(2)} \end{bmatrix} = \begin{bmatrix} f_{x2}^{(2)} \\ f_{y2}^{(2)} \\ f_{x3}^{(2)} \\ f_{y3}^{(2)} \end{bmatrix}$$

(1) Assemble the master stiffness equations.

(2) Apply the given force and displacement BCs to get a reduced system of 2 equations and show it.

(3) Solve the reduced stiffness system for the unknown displacements and show the complete node displacement vector.

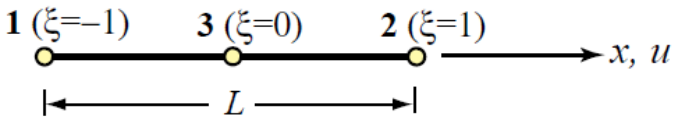
(4) Recover the axial force $F^{(2)}$ in element (2) using the displacements you got in (3), noting sign.



2. (40 pts) The isoparametric definition of a straight 3-node bar element, pictured below, in its local system is

$$\begin{Bmatrix} 1 \\ x \\ u \end{Bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ u_1 & u_2 & u_3 \end{bmatrix} \begin{Bmatrix} N_1^e(\xi) \\ N_2^e(\xi) \\ N_3^e(\xi) \end{Bmatrix}$$

where $u = u(x)$ is the axial displacement, $N_i^e(\xi)$ are the shape functions, and $\xi$ is an isoparametric coordinate equal to $-1$, 1 and 0 at nodes 1, 2 and 3, respectively. For all items below, take the $x$ node coordinates to be $x_1 = 0$, $x_2 = L$ and $x_3 = L/2$, so that node 3 is at the midpoint of 1–2.




(1) Determine the shape functions.

(2) Show that the Jacobian matrix (here just a scalar) $J = dx/d\xi$ reduces to $L/2$ at any point in the element.

(3) A uniform distributed force $q$ (force per unit length) acts along the longitudinal direction $x$. The energy-consistent node force vector is given by $\mathbf{f}^e = \int_0^L q\mathbf{N}^T dx$ where the 3x1 matrix $\mathbf{N}^T$ collects the 3 shape functions given above. Show that each corner gets only 1/6 of the total load $qL$ whereas the midpoint gets 2/3.

(4) Suppose the integral in (3) is done by Gauss quadrature. How many Gauss points (of a *p*-point one-dimensional rule) would be needed to get the same analytical answer? Explain your answer but don't do any computations.

(5) Could the shape functions be used to build a 3-node Bernoulli-Euler *plane beam* element in which *u* is the *transverse* (not the axial) displacement? Explain your answer but do not write formulas.

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3. (30 pts) The exam questions pertain to the development and implementation of a finite element model to numerically solve the two-dimensional Poisson equation over a plane domain $\Omega$ : $k\nabla^2 u = k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = s$ in $\Omega$ Here $\nabla^2$ is the Laplacian, $u = u(x, y)$ is a scalar unknown function, $k$ a known constitutive coefficient that may be function of $x$ and $y$, and $s = s(x, y)$ is a given source function. The actual body has a thickness $h$ in the $z$ direction; often this thickness is uniform and taken equal to unity. The Poisson partial differential equation models many important problems in mathematical physics. Some of them are: steady-state heat conduction, St-Venant torsion of arbitrary cross sections, potential fluid flow, linear acoustics, hydrostatics and electrostatics. The physical meaning of $u$ (as well as of that of $k$ and $s$) depends on the application. In the *heat conduction* problem considered here, $u$ is the temperature, $k$ the coefficient of thermal conductivity, $q_n = -k(\partial u/\partial n)$ the heat flux along a direction $n$ and $s$ the internal heat source density (heat produced per unit volume in the material). To finish the problem specification, the Poisson equation has to be complemented by boundary conditions (BC) on the domain boundary $\Gamma$. The classical BC are of two types. Over a portion $\Gamma_1$ of the domain boundary $\Gamma$ (distinguished by a dashed line in the figure) the value of $u$ is prescribed to be $\bar{u}$. Over the complementary portion $\Gamma_2$ the flux $q_n = -k(\partial u/\partial n)$, where $n$ is the *exterior* normal to $\Gamma$, is prescribed to be $\bar{q}_n$: $u = \bar{u}$ on $\Gamma_1$, $q_n = -k\frac{\partial u}{\partial n} = \bar{q}_n$ on $\Gamma_2$. The heat flux is considered positive if heat flows away from $\Omega$. In the mathematical literature the conditions on $\Gamma_1$ and $\Gamma_2$ are referred to as Dirichlet and Neumann boundary conditions, respectively.

(1) A variational form equivalent to Poisson equation and BC is $\delta\Pi = 0$, where $\delta$ denotes variation with respect to the unknown function $u$, and $\Pi$ is the total energy functional $\Pi(u) = U(u) - W(u)$ where $U$ and $W$ represent internal energy and external potential, respectively. With respect to this variational principle the Dirichlet $u = \bar{u}$ on $\Gamma_1$ is essential, whereas the Neumann BC $q_n = -k\frac{\partial u}{\partial n} = \bar{q}_n$ on $\Gamma_2$ is natural. Determine $U$ and $W$.

(2) The domain $\Omega$ is discretized with 4-node bilinear-quadrilateral finite elements as sketched in the figure. Over a generic quadrilateral 1-2-3-4, the temperature $u$ is approximated by the bilinear iso-P interpolation:

$$u^e = N_1 u_1 + N_2 u_2 + N_3 u_3 + N_4 u_4 = \begin{bmatrix} N_1 & N_2 & N_3 & N_4 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \mathbf{N}\mathbf{u}^e$$

where $u_1$, $u_2$, $u_3$ and $u_4$ are the node temperatures, and $N_1$, $N_2$, $N_3$ and $N_4$ are the usual iso-P shape functions for the 4-node quadrilateral. Determine four shape functions.

(3) The temperature gradients $g_x = \partial u/\partial x$ and $g_y = \partial u/\partial y$ are mathematically analogous to strains in mechanical problems. We call $\mathbf{g}^T = \begin{bmatrix} g_x & g_y \end{bmatrix}$ the thermal gradient vector and the matrix $\mathbf{B}$ that relates $\mathbf{g}$ to the element node

values $\mathbf{g}=\begin{bmatrix} g_x \\ g_y \end{bmatrix}=\begin{bmatrix} \partial u/\partial x \\ \partial u/\partial y \end{bmatrix}=\begin{bmatrix} \partial N_1/\partial x & \partial N_2/\partial x & \partial N_3/\partial x & \partial N_4/\partial x \\ \partial N_1/\partial y & \partial N_2/\partial y & \partial N_3/\partial y & \partial N_4/\partial y \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}=\mathbf{B}\mathbf{u}^e$ the thermal-gradient-to-temperature matrix. The first variation $\delta\Pi^e=\left(\delta\mathbf{u}^e\right)\left(\mathbf{K}^e\mathbf{u}^e-\mathbf{f}^e\right)=0$ yields the element equations $\mathbf{K}^e\mathbf{u}^e=\mathbf{f}^e$. Here $\mathbf{K}^e$ is $4\times4$ element stiffness matrix, which derives from the element internal energy $U^e$, and $\mathbf{f}^e$ is the element node force vector, which derives from the element external potential $W^e$. Show that $\mathbf{K}^e=\int_{\Omega^e} kh\mathbf{B}^T\mathbf{B}\,d\Omega^e$ and check that for a *rectangular element* of sizes $a\times b$ along the *x* and *y* axes, respectively, this gives

$$\mathbf{K}^e=\frac{1}{6}kh\begin{bmatrix} \frac{2a}{b}+\frac{2b}{a} & \frac{a}{b}-\frac{2b}{a} & -\frac{a}{b}-\frac{b}{a} & -\frac{2a}{b}+\frac{b}{a} \\ & \frac{2a}{b}+\frac{2b}{a} & -\frac{2a}{b}+\frac{b}{a} & -\frac{a}{b}-\frac{b}{a} \\ & & \frac{2a}{b}+\frac{2b}{a} & \frac{a}{b}-\frac{2b}{a} \\ sym & & & \frac{2a}{b}+\frac{2b}{a} \end{bmatrix}$$

Both *k* and *h* can be assumed to be constant over the element.

(4) Why is a 2 x 2 Gauss integration rule used for $\mathbf{K}^e$ of the 4-node quad? Wouldn't 1 x 1 be enough? (Be careful as regards the $n_E$ and $n_R$ values.)

(5) If a 6-node triangle element and an 8-node quadrilateral elements were to be developed for this problem, how many Gauss points would be needed so that $\mathbf{K}^e$ is rank sufficient?