Final Exam

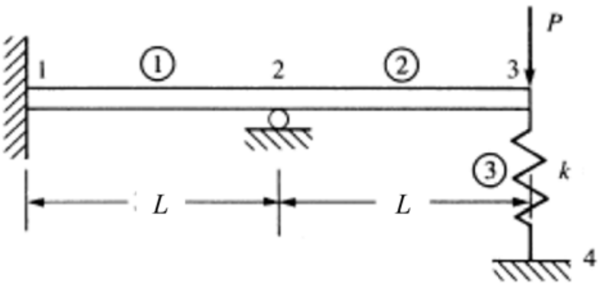
12/18/2018

1. (10 pts) Derive the equivalent spring formula $F = \left(\frac{EA}{L}\right)d$ by the Theory of Elasticity relations $e = \frac{d\bar{u}(\bar{x})}{d\bar{x}}$ (strain-displacement equation), $\sigma = Ee$ (Hooke's law) and $F = A\sigma$ (axial force definition). Here e is the axial strain (independent of $\bar{x}$) and σ the axial stress (also independent of $\bar{x}$). Finally, $\bar{u}(\bar{x})$ denotes the axial displacement of the cross section at a distance $\bar{x}$ from node i, which is linearly interpolated as $\bar{u}(\bar{x}) = \bar{u}_{xi}\left(1-\frac{\bar{x}}{L}\right)+\bar{u}_{xj}\frac{\bar{x}}{L}$. Justify that $\bar{u}(\bar{x})$ is correct since the bar differential equilibrium equation: $\frac{d}{d\bar{x}}\left[A\left(\frac{d\sigma}{d\bar{x}}\right)\right]=0$, is verified for all $\bar{x}$ if A is constant along the bar.

2. (20 pts) We have discretized the beam as shown by the node numbering. The beam is fixed at node 1, has a roller support at node 2, and has an elastic spring support at node 3.

(1) Derive the master stiffness equation.

(2) Derive the reduced master stiffness equation by applying boundary conditions.



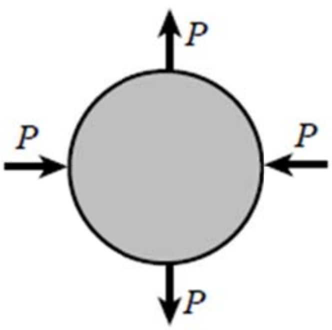
Element stiffness of a beam:

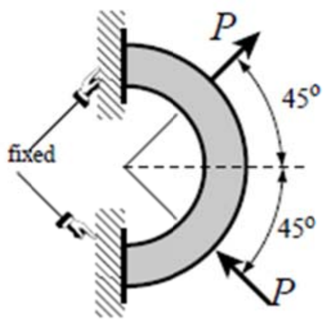
$$\mathbf{K}^e = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ & 4L^2 & -6L & 2L^2 \\ & & 12 & -6L \\ sym & & & 4L^2 \end{bmatrix}$$

3. (20 pts) Consider the two-dimensional problems illustrated in the figures.

(1) Identify all the symmetry and anti-symmetry lines.

(2) Draw a coarse FE mesh indicating, with rollers or fixed supports, which kind of displacement BCs you would specify on the symmetry or anti-symmetry lines.





4. (20 pts) The simplest triangular element for plane stress is the three-node triangle with *linear shape functions*, with degrees of freedom located at the corners.

(1) Write down the linear interpolation for the displacement components u_x and u_y at an arbitrary point *P*.

(2) Write down the strain-displacement equations and the stress-strain equations. [Note that the strains are *constant* over the element. This is the origin of the name *constant strain triangle* constant strain triangle (CST) given it in many finite element publications.]

(3) Derive the stiffness matrix of Turner triangle with uniform thickness h.

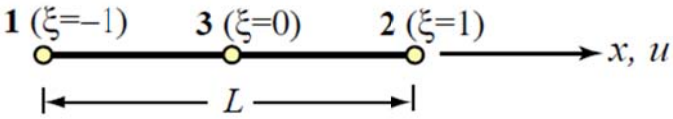
(4) Derive the consistent nodal force vector when only internal body forces are considered.

[Hint: $\begin{bmatrix} 1 \\ x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \end{bmatrix} \to \begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \end{bmatrix} = \frac{1}{2A} \begin{bmatrix} x_2 y_3 - x_3 y_2 & y_2 - y_3 & x_3 - x_2 \\ x_3 y_1 - x_1 y_3 & y_3 - y_1 & x_1 - x_3 \\ x_1 y_2 - x_2 y_1 & y_1 - y_2 & x_2 - x_1 \end{bmatrix} \begin{bmatrix} 1 \\ x \\ y \end{bmatrix}$]

5. (30 pts) The isoparametric definition of a straight 3-node bar element, pictured below, in its local system is

$$\begin{Bmatrix} 1 \\ x \\ u \end{Bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ u_1 & u_2 & u_3 \end{bmatrix} \begin{Bmatrix} N_1^e(\xi) \\ N_2^e(\xi) \\ N_3^e(\xi) \end{Bmatrix}$$

where $u = u(x)$ is the axial displacement, $N_i^e(\xi)$ are the shape functions, and ξ is an isoparametric coordinate equal to −1, 1 and 0 at nodes 1, 2 and 3, respectively. For all items below, take the x node coordinates to be $x_1 = 0,\ x_2 = L$ and $x_3 = L/2$, so that node 3 is at the midpoint of 1–2.



(1) Determine the shape functions.

(2) Show that the Jacobian matrix (here just a scalar) $J = dx/d\xi$ reduces to $L/2$ at any point in the element.

(3) A uniform distributed force *q* (force per unit length) acts along the longitudinal direction *x*. The energy-consistent node force vector is given by $\mathbf{f}^e = \int_0^L q\mathbf{N}^T dx$ where the 3x1 matrix $\mathbf{N}^T$ collects the 3 shape functions given above. Show that each corner gets only 1/6 of the total load *qL* whereas the midpoint gets 2/3.

(4) Suppose the integral in (3) is done by Gauss quadrature. How many Gauss points (of a *p*-point one-dimensional rule) would be needed to get the same analytical answer? Explain your answer but don't do any computations.