Final Exam

12/19/2017

1. (30 pts)

(1) $$\begin{bmatrix} 50 & -50 & -50 & 50 & 0 & 0 \\ -50 & 50 & 50 & -50 & 0 & 0 \\ -50 & 50 & 100 & 0 & -50 & -50 \\ 50 & -50 & 0 & 100 & -50 & -50 \\ 0 & 0 & -50 & -50 & 50 & 50 \\ 0 & 0 & -50 & -50 & 50 & 50 \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \end{bmatrix} = \begin{bmatrix} f_{x1} \\ f_{y1} \\ f_{x2} \\ f_{y2} \\ f_{x3} \\ f_{y3} \end{bmatrix} \text{(5 pts)}$$

(2) $$\begin{bmatrix} 50 & -50 & -50 & 50 & 0 & 0 \\ -50 & 50 & 50 & -50 & 0 & 0 \\ -50 & 50 & 100 & 0 & -50 & -50 \\ 50 & -50 & 0 & 100 & -50 & -50 \\ 0 & 0 & -50 & -50 & 50 & 50 \\ 0 & 0 & -50 & -50 & 50 & 50 \end{bmatrix} \begin{bmatrix} 0 \\ u_{y1} \\ u_{x2} \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} f_{x1} \\ 5 \\ 0 \\ f_{y2} \\ f_{x3} \\ f_{y3} \end{bmatrix} \rightarrow \begin{bmatrix} 50 & 50 \\ 50 & 100 \end{bmatrix} \begin{bmatrix} u_{y1} \\ u_{x2} \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \end{bmatrix} \text{(10 pts)}$$

(3) $$\begin{bmatrix} u_{y1} \\ u_{x2} \end{bmatrix} = \begin{bmatrix} -1/10 \\ 1/5 \end{bmatrix} \rightarrow \mathbf{u} = \begin{bmatrix} 0 & 0.2 & -0.1 & 0 & 0 & 0 \end{bmatrix}^T \text{(5 pts)}$$

(4) (10 pts) The orientation angle from x to $2 \rightarrow 3$ (+CCW) is 45. We have $c = \cos 45° = 1/\sqrt{2}$ and $s = \sin 45° = 1/\sqrt{2}$. The local displacements are recovered from the displacement transformation

$$\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 & 0 \\ 0 & 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} u_{x2} = -0.1 \\ u_{y2} = 0 \\ u_{x3} = 0 \\ u_{y3} = 0 \end{bmatrix} = \begin{bmatrix} \bar{u}_{x2} = -0.1/\sqrt{2} \\ * \\ \bar{u}_{x3} = 0 \\ * \end{bmatrix}$$

where * are values of no interest for this computation. The member elongation is $d^{(2)} = \bar{u}_{x3} - \bar{u}_{x2} = 0.1/\sqrt{2}$. (5 pts)

$$p^{(2)} = \frac{E^{(2)} A^{(2)}}{L^{(2)}} d^{(2)} = \frac{600\sqrt{2}}{6\sqrt{2}} \left(\frac{1}{10\sqrt{2}} \right) = \frac{10}{\sqrt{2}} = +7.07 (\text{tension}) \text{(5 pts)} \quad (w/o \text{ sign -2 pts})$$

2. (40 pts)

(1) $N_1^e = -\frac{1}{2}\xi(1-\xi), N_2^e = \frac{1}{2}\xi(1+\xi), N_3^e = 1-\xi^2$ (3 pts each)

(2) $x = x_1 N_1^e + x_2 N_2^e + x_3 N_3^e = \frac{1}{2}L(\xi+1) \rightarrow J = \frac{dx}{d\xi} = \frac{L}{2}$ (6 pts)

(3)
$$\begin{cases} f_1 = q\int_{-1}^{1} \frac{1}{2}\xi(1-\xi)\left(\frac{L}{2}\right)d\xi = \frac{1}{4}qL\left[\frac{1}{3}\xi^3 - \frac{1}{2}\xi^2\right]_{-1}^{1} = \frac{1}{4}qL\left(\frac{2}{3}\right) = \frac{1}{6}qL \text{ (6 pts)} \\ f_3 = q\int_{-1}^{1} (1-\xi^2)\left(\frac{L}{2}\right)d\xi = \frac{1}{2}qL\left[\xi - \frac{1}{3}\xi^3\right]_{-1}^{1} = \frac{1}{2}qL\left(\frac{4}{3}\right) = \frac{2}{3}qL \text{ (6 pts)} \end{cases}$$

For other corner, $f_1 = f_2$ on account of symmetry. Check: $\left(\frac{1}{6}+\frac{1}{6}+\frac{2}{3}\right)qL = qL$ (Bonus 3 pts)

(4) The integrand is quadratic in ξ if J is constant because q is constant and the shape functions in N are quadratic in ξ. Two Gauss points would be enough because a 2-point rule integrates exactly up to cubics, being exact up to polynomial order $2\times 2-1=3$.(5 pts)

(5) Bernoulli-Euler beam elements must have C^1 interelement continuity because second derivatives (curvatures u'') appear in the TPE functional, giving a variational index of $m = 2$. For a 1D element this is verified if the transverse displacement u and the slope u' are continuous at the end nodes. But a 3-node beam element, with the given shape functions defining $u(\xi)$, would have only u_1, u_2 and u_3 as degrees of freedom, missing $\theta_1 = u_1'$ and $\theta_2 = u_2'$. This guarantees only C_0 continuity. Thus this choice would be *unsuitable* for constructing a Bernoulli-Euler beam element. (8 pts)

3. (30 pts)

(1) $$\begin{cases} U(u) = \frac{1}{2}\int_{\Omega} kh\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right] d\Omega \ \text{(4 pts)} \\ W(u) = \int_{\Omega} shud\Omega - \oint_{\Gamma_2} \bar{q}_n hud\Gamma \ \text{(4 pts)} \end{cases}$$

(2) $$\begin{cases} N_1^e = \frac{1}{4}(1-\xi)(1-\eta) \\ N_2^e = \frac{1}{4}(1+\xi)(1-\eta) \\ N_3^e = \frac{1}{4}(1+\xi)(1+\eta) \\ N_4^e = \frac{1}{4}(1-\xi)(1+\eta) \end{cases} \text{(1 pt each)} \rightarrow \begin{cases} \frac{\partial N_1^e}{\partial \xi} = -\frac{1}{4}(1-\eta), \frac{\partial N_1^e}{\partial \eta} = -\frac{1}{4}(1-\xi) \\ \frac{\partial N_2^e}{\partial \xi} = \frac{1}{4}(1-\eta), \frac{\partial N_2^e}{\partial \eta} = -\frac{1}{4}(1+\xi) \\ \frac{\partial N_3^e}{\partial \xi} = \frac{1}{4}(1+\eta), \frac{\partial N_3^e}{\partial \eta} = \frac{1}{4}(1+\xi) \\ \frac{\partial N_4^e}{\partial \xi} = -\frac{1}{4}(1+\eta), \frac{\partial N_4^e}{\partial \eta} = \frac{1}{4}(1-\xi) \end{cases}$$

(3) $(x_1, y_1) = (0,0)$ $(x_2, y_2) = (a,0)$ $(x_3, y_3) = (a,b)$ $(x_4, y_4) = (0,b)$

$$\mathbf{J} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} = \begin{bmatrix} \sum_i^n x_i \frac{\partial N_i^e}{\partial \xi} & \sum_i^n y_i \frac{\partial N_i^e}{\partial \xi} \\ \sum_i^n x_i \frac{\partial N_i^e}{\partial \eta} & \sum_i^n y_i \frac{\partial N_i^e}{\partial \eta} \end{bmatrix} = \begin{bmatrix} \frac{a}{2} & 0 \\ 0 & \frac{b}{2} \end{bmatrix} \rightarrow J = \frac{ab}{4} \rightarrow \mathbf{J}^{-1} = \frac{4}{ab}\begin{bmatrix} \frac{b}{2} & 0 \\ 0 & \frac{a}{2} \end{bmatrix} = 2\begin{bmatrix} \frac{1}{a} & 0 \\ 0 & \frac{1}{b} \end{bmatrix} = \begin{bmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \eta}{\partial x} \\ \frac{\partial \xi}{\partial y} & \frac{\partial \eta}{\partial y} \end{bmatrix} \text{(4 pts)}$$

$$\mathbf{B} = \begin{bmatrix} \frac{\partial N_1^e}{\partial x} & \frac{\partial N_2^e}{\partial x} & \frac{\partial N_3^e}{\partial x} & \frac{\partial N_4^e}{\partial x} \\ \frac{\partial N_1^e}{\partial y} & \frac{\partial N_2^e}{\partial y} & \frac{\partial N_3^e}{\partial y} & \frac{\partial N_4^e}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{\partial N_1^e}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial N_1^e}{\partial \eta}\frac{\partial \eta}{\partial x} & \frac{\partial N_2^e}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial N_2^e}{\partial \eta}\frac{\partial \eta}{\partial x} & \frac{\partial N_3^e}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial N_3^e}{\partial \eta}\frac{\partial \eta}{\partial x} & \frac{\partial N_4^e}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial N_4^e}{\partial \eta}\frac{\partial \eta}{\partial x} \\ \frac{\partial N_1^e}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial N_1^e}{\partial \eta}\frac{\partial \eta}{\partial y} & \frac{\partial N_2^e}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial N_2^e}{\partial \eta}\frac{\partial \eta}{\partial y} & \frac{\partial N_3^e}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial N_3^e}{\partial \eta}\frac{\partial \eta}{\partial y} & \frac{\partial N_4^e}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial N_4^e}{\partial \eta}\frac{\partial \eta}{\partial y} \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{4}(1-\eta)\frac{2}{a} & \frac{1}{4}(1-\eta)\frac{2}{a} & \frac{1}{4}(1+\eta)\frac{2}{a} & -\frac{1}{4}(1+\eta)\frac{2}{a} \\ -\frac{1}{4}(1-\xi)\frac{2}{b} & -\frac{1}{4}(1+\xi)\frac{2}{b} & \frac{1}{4}(1+\xi)\frac{2}{b} & \frac{1}{4}(1-\xi)\frac{2}{b} \end{bmatrix} \text{(4 pts)}$$

$$\mathbf{K}^e = \int_{\Omega^e} kh\mathbf{B}^T\mathbf{B}d\Omega^e = kh\int_{-1}^{+1}\int_{-1}^{+1}\mathbf{B}^T\mathbf{B}\det\mathbf{J}d\xi d\eta$$

$$\mathbf{K}_{11}^e = kh\left(\frac{ab}{4}\right)\int_{-1}^{+1}\int_{-1}^{+1}\left[\frac{1}{4a^2}(1-\eta)^2 + \frac{1}{4b^2}(1-\xi)^2\right]d\xi d\eta = kh\left(\frac{1}{3}\frac{b}{a} + \frac{1}{3}\frac{a}{b}\right)$$

$$\mathbf{K}_{12}^e = kh\left(\frac{ab}{4}\right)\int_{-1}^{+1}\int_{-1}^{+1}\left[-\frac{1}{4a^2}(1-\eta)^2 + \frac{1}{4b^2}(1-\xi)(1+\xi)\right]d\xi d\eta = kh\left(-\frac{1}{3}\frac{b}{a} + \frac{1}{6}\frac{a}{b}\right)$$

$$\mathbf{K}_{13}^e = kh\left(\frac{ab}{4}\right)\int_{-1}^{+1}\int_{-1}^{+1}\left[-\frac{1}{4a^2}(1-\eta)(1+\eta) - \frac{1}{4b^2}(1-\xi)(1+\xi)\right]d\xi d\eta = kh\left(-\frac{1}{6}\frac{b}{a} - \frac{1}{6}\frac{a}{b}\right)$$

$$\mathbf{K}_{14}^e = kh\left(\frac{ab}{4}\right)\int_{-1}^{+1}\int_{-1}^{+1}\left[\frac{1}{4a^2}(1-\eta)(1+\eta) - \frac{1}{4b^2}(1-\xi)^2\right]d\xi d\eta = kh\left(\frac{1}{6}\frac{b}{a} - \frac{1}{3}\frac{a}{b}\right)$$

$$\mathbf{K}^e_{22} = kh\left(\frac{ab}{4}\right)\int_{-1}^{+1}\int_{-1}^{+1}\left[\frac{1}{4a^2}(1-\eta)^2 + \frac{1}{4b^2}(1+\xi)^2\right]d\xi d\eta = kh\left(\frac{1}{3}\frac{b}{a}+\frac{1}{3}\frac{a}{b}\right)$$

$$\mathbf{K}^e_{23} = kh\left(\frac{ab}{4}\right)\int_{-1}^{+1}\int_{-1}^{+1}\left[\frac{1}{4a^2}(1-\eta)(1+\eta) - \frac{1}{4b^2}(1+\xi)^2\right]d\xi d\eta = kh\left(\frac{1}{6}\frac{b}{a}-\frac{1}{3}\frac{a}{b}\right)$$

$$\mathbf{K}^e_{24} = kh\left(\frac{ab}{4}\right)\int_{-1}^{+1}\int_{-1}^{+1}\left[-\frac{1}{4a^2}(1-\eta)(1+\eta) - \frac{1}{4b^2}(1+\xi)(1-\xi)\right]d\xi d\eta = kh\left(-\frac{1}{6}\frac{b}{a}-\frac{1}{6}\frac{a}{b}\right)$$

$$\mathbf{K}^e_{33} = kh\left(\frac{ab}{4}\right)\int_{-1}^{+1}\int_{-1}^{+1}\left[\frac{1}{4a^2}(1+\eta)^2 + \frac{1}{4b^2}(1+\xi)^2\right]d\xi d\eta = kh\left(\frac{1}{3}\frac{b}{a}+\frac{1}{3}\frac{a}{b}\right)$$

$$\mathbf{K}^e_{34} = kh\left(\frac{ab}{4}\right)\int_{-1}^{+1}\int_{-1}^{+1}\left[\frac{1}{4a^2}(1+\eta)^2 + \frac{1}{4b^2}(1+\xi)(1-\xi)\right]d\xi d\eta = kh\left(-\frac{1}{3}\frac{b}{a}+\frac{1}{6}\frac{a}{b}\right)$$

$$\mathbf{K}^e_{44} = kh\left(\frac{ab}{4}\right)\int_{-1}^{+1}\int_{-1}^{+1}\left[\frac{1}{4a^2}(1+\eta)^2 + \frac{1}{4b^2}(1-\xi)^2\right]d\xi d\eta = kh\left(\frac{1}{3}\frac{b}{a}+\frac{1}{3}\frac{a}{b}\right)$$

$$\mathbf{K}^e == \frac{1}{6}kh\begin{bmatrix} \frac{2a}{b}+\frac{2b}{a} & \frac{a}{b}-\frac{2b}{a} & -\frac{a}{b}-\frac{b}{a} & -\frac{2a}{b}+\frac{b}{a} \\ & \frac{2a}{b}+\frac{2b}{a} & -\frac{2a}{b}+\frac{b}{a} & -\frac{a}{b}-\frac{b}{a} \\ & & \frac{2a}{b}+\frac{2b}{a} & \frac{a}{b}-\frac{2b}{a} \\ sym & & & \frac{2a}{b}+\frac{2b}{a} \end{bmatrix}$$ (한 컴포넌트라도 맞으면 보너스 5 pts)

(4)

The number of DOFs per node is 1.(2 pts)

In 2D, the dimension of the constitutive matrix is 2 (it connects two temperature gradients to two fluxes).

$\rightarrow n_E = 2$ (2 pts)

The number of rigid body modes is 1. $\rightarrow n_R = 1$ (2 pts)

A "rigid body mode" in this context is a a vector $\mathbf{u}^e_R$ of equal temperatures at each node.

If the 4 node temperatures are equal, the whole element is at constant temperature.

Postmultiplying $\mathbf{K}^e\mathbf{u}^e_R = 0$ because a constant temperature state produces no heat fluxes: the gradients are zero.

This is analogous to the definition of rigid body mode in structures.

n	n_F	n_R	n_E	$\min n_G\left[n_E n_G \ge (n_F - n_R)\right]$	
4	4	1	2	2	(2 pts)

(5)

n	n_F	n_R	n_E	$\min n_G\left[n_E n_G \ge (n_F - n_R)\right]$	
6	6	1	2	3	(1 pts)
8	8	1	2	4	(1 pts)