# Final Exam

12/18/2018

1.

$e = \frac{u_{xj} - u_{xi}}{L} = \frac{d}{L} \rightarrow \sigma = Ee = E\frac{d}{L} \rightarrow F = A\sigma = AE\frac{d}{L} = k_s d$ where $k_s = \frac{EA}{L}$ (5 pts)

The bar equation is satisfied by a constant $A\sigma$. Because $A$ is constant so are the axial stress $\sigma$ and axial strain $e = \frac{\sigma}{E} = \frac{d\bar{u}(\bar{x})}{d\bar{x}}$.

Therefore the displacement $\bar{u}$ must vary linearly in $\bar{x}$. (5 pts)

2.

beam의 자유도: $(x, y)$가 아니라 $(v, \theta)$임 $\rightarrow$ 대부분 오답

$$\mathbf{K}^{(1)} = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ & 4L^2 & -6L & 2L^2 \\ & & 12 & -6L \\ sym & & & 4L^2 \end{bmatrix}, \mathbf{K}^{(2)} = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ & 4L^2 & -6L & 2L^2 \\ & & 12 & -6L \\ sym & & & 4L^2 \end{bmatrix}, \mathbf{K}^{(3)} = k\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$ (스프링 고려한 경우, 7 pts)

$$\frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L & 0 & 0 & 0 \\ & 4L^2 & -6L & 2L^2 & 0 & 0 & 0 \\ & & 12+12 & -6L+6L & 12 & 6L & 0 \\ & & & 4L^2+4L^2 & -6L & 2L^2 & 0 \\ & & & & 12+k\frac{L^3}{EI} & -6L & -k\frac{L^3}{EI} \\ & & & & & 4L^2 & 0 \\ sym & & & & & & k\frac{L^3}{EI} \end{bmatrix}\begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \\ v_3 \\ \theta_3 \\ v_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -P \\ 0 \\ 0 \end{Bmatrix}$$

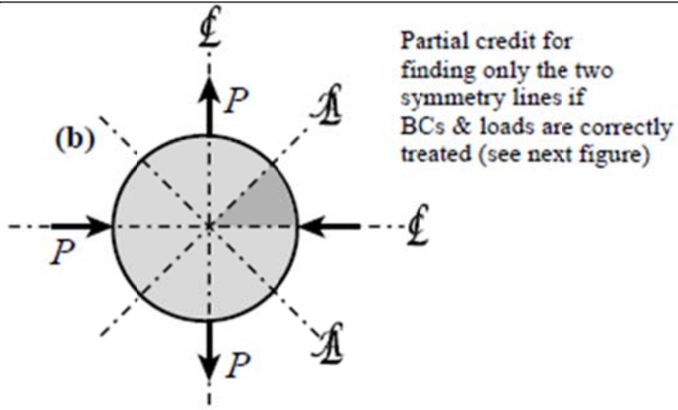
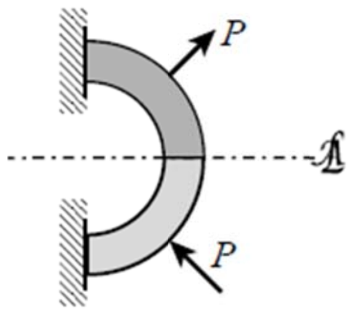
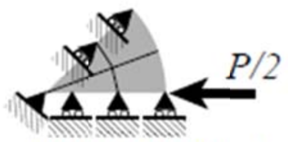
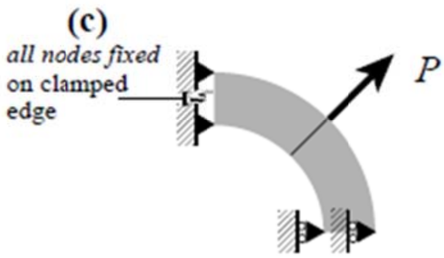
$$\rightarrow \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L & 0 & 0 & 0 \\ & 4L^2 & -6L & 2L^2 & 0 & 0 & 0 \\ & & 24 & 0 & 12 & 6L & 0 \\ & & & 8L^2 & -6L & 2L^2 & 0 \\ & & & & 12+k\frac{L^3}{EI} & -6L & -k\frac{L^3}{EI} \\ & & & & & 4L^2 & 0 \\ sym & & & & & & k\frac{L^3}{EI} \end{bmatrix}\begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \\ v_3 \\ \theta_3 \\ v_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -P \\ 0 \\ 0 \end{Bmatrix}$$ (8 pts)

요소1과 2의 어셈블인 경우(4 pts), 방정식이 아니면 −1 pts

(2) $v_1 = \theta_1 = v_2 = v_4 = 0$ (5 pts)

$$\frac{EI}{L^3}\begin{bmatrix} 8L^2 & -6L & 2L^2 \\ & 12+k\frac{L^3}{EI} & -6L \\ sym & & 4L^2 \end{bmatrix}\begin{Bmatrix} \theta_2 \\ v_3 \\ \theta_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -P \\ 0 \end{Bmatrix}$$

3. (5 pts each) 요소절점에 경계조건과 하중조건이 작용되어야 함

| | |
|---|---|
|  |  |
| <br>Full credit for this solution<br>하중 2점, 대칭/비대칭/가운데 고정 각 1점 | <br>하중 1점, 대칭/비대칭 각 2점 |

4. (1) (2 pts)

$$\begin{bmatrix} u_x \\ u_y \end{bmatrix} = \begin{bmatrix} \zeta_1 & 0 & \zeta_2 & 0 & \zeta_3 & 0 \\ 0 & \zeta_1 & 0 & \zeta_2 & 0 & \zeta_3 \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \end{bmatrix} = \underbrace{\mathbf{N}}_{\substack{\text{shape functions} \\ N_i=\zeta_i,\ i=1,2,3}} \mathbf{u}^e$$

(2)

$$\begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \end{bmatrix} = \frac{1}{2A} \begin{bmatrix} x_2 y_3 - x_3 y_2 & y_2 - y_3 & x_3 - x_2 \\ x_3 y_1 - x_1 y_3 & y_3 - y_1 & x_1 - x_3 \\ x_1 y_2 - x_2 y_1 & y_1 - y_2 & x_2 - x_1 \end{bmatrix} \begin{bmatrix} 1 \\ x \\ y \end{bmatrix} \rightarrow \begin{bmatrix} \dfrac{\partial f}{\partial x} \\ \dfrac{\partial f}{\partial y} \end{bmatrix} = \frac{1}{2A} \begin{bmatrix} y_{23} & y_{31} & y_{12} \\ x_{32} & x_{13} & x_{21} \end{bmatrix} \begin{bmatrix} \dfrac{\partial f}{\partial \zeta_1} \\ \dfrac{\partial f}{\partial \zeta_2} \\ \dfrac{\partial f}{\partial \zeta_3} \end{bmatrix} \quad (4\text{ pts})$$

$$\mathbf{e} = \mathbf{DNu^e} = \frac{1}{2A} \begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \end{bmatrix} = \mathbf{Bu^e} \quad (2\text{ pts})$$

$$\sigma = \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} = \begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{12} & E_{22} & E_{23} \\ E_{13} & E_{23} & E_{33} \end{bmatrix} \begin{bmatrix} e_{xx} \\ e_{yy} \\ 2e_{xy} \end{bmatrix} = \mathbf{Ee} \quad (2\text{ pts})$$

(3)

$$\mathbf{K}^e = \int_{\Omega^e} h\mathbf{B}^T\mathbf{EB}\, d\Omega^e \xrightarrow[\text{over the triangle area}]{\mathbf{B}\text{ and }\mathbf{E}\text{ are constant}} \mathbf{K}^e = \mathbf{B}^T\mathbf{EB}\int_{\Omega^e} h\, d\Omega^e \quad (2\text{ pts})$$

$$= \frac{1}{4A^2} \begin{bmatrix} y_{23} & 0 & x_{32} \\ 0 & x_{32} & y_{23} \\ y_{31} & 0 & x_{13} \\ 0 & x_{13} & y_{31} \\ y_{12} & 0 & x_{21} \\ 0 & x_{21} & y_{12} \end{bmatrix} \begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{12} & E_{22} & E_{23} \\ E_{13} & E_{23} & E_{33} \end{bmatrix} \begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix} \int_{\Omega^e} h\, d\Omega^e \quad (2\text{ pts})$$

$$\xrightarrow{\text{if } h \text{ is constant}} \mathbf{K}^e = \frac{h}{4A} \begin{bmatrix} y_{23} & 0 & x_{32} \\ 0 & x_{32} & y_{23} \\ y_{31} & 0 & x_{13} \\ 0 & x_{13} & y_{31} \\ y_{12} & 0 & x_{21} \\ 0 & x_{21} & y_{12} \end{bmatrix} \begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{12} & E_{22} & E_{23} \\ E_{13} & E_{23} & E_{33} \end{bmatrix} \begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix} \quad (1\text{ pt})$$

(4)

$$\mathbf{f}^e = \int_{\Omega^e} h\mathbf{N}^T\mathbf{b}\,d\Omega^e = \int_{\Omega^e} h\begin{bmatrix}\zeta_1 & 0\\ 0 & \zeta_1\\ \zeta_2 & 0\\ 0 & \zeta_2\\ \zeta_3 & 0\\ 0 & \zeta_3\end{bmatrix}\begin{bmatrix}b_x\\ b_y\end{bmatrix}d\Omega^e \;(3\text{ pts}) \xrightarrow[\text{over the element}]{\text{if body forces and } h \text{ are constant}} \mathbf{f}^e = \frac{1}{3}Ah\underbrace{\begin{bmatrix}b_x\\ b_y\\ b_x\\ b_y\\ b_x\\ b_y\end{bmatrix}}_{\text{same as "load limping"}} \;(2\text{ pts})$$

5.

$(1)\; N_1^e = -\frac{1}{2}\xi(1-\xi),\, N_2^e = \frac{1}{2}\xi(1+\xi),\, N_3^e = 1-\xi^2$ (3 pts each)

$(2)\; x = x_1N_1^e + x_2N_2^e + x_3N_3^e = \frac{1}{2}L(\xi+1) \rightarrow J = \frac{dx}{d\xi} = \frac{L}{2}$ (4 pts)

$$(3)\;\begin{cases} f_1 = q\int_{-1}^{1}\frac{1}{2}\xi(1-\xi)\left(\frac{L}{2}\right)d\xi = \frac{1}{4}qL\left[\frac{1}{3}\xi^3 - \frac{1}{2}\xi^2\right]_{-1}^{1} = \frac{1}{4}qL\left(\frac{2}{3}\right) = \frac{1}{6}qL \;(6\text{ pts})\\ f_3 = q\int_{-1}^{1}(1-\xi^2)\left(\frac{L}{2}\right)d\xi = \frac{1}{2}qL\left[\xi - \frac{1}{3}\xi^3\right]_{-1}^{1} = \frac{1}{2}qL\left(\frac{4}{3}\right) = \frac{2}{3}qL \;(6\text{ pts})\end{cases}$$

For other corner, $f_1 = f_2$ on account of symmetry. Check: $\left(\frac{1}{6}+\frac{1}{6}+\frac{2}{3}\right)qL = qL$

$(4)$ The integrand is quadratic in $\xi$ if $J$ is constant because $q$ is constant and the shape functions in $N$ are quadratic in $\xi$. Two Gauss points would be enough because a 2-point rule integrates exactly up to cubics, being exact up to polynomial order $2\times 2-1=3$. (5 pts)

OR $n_G n_E \geq n_F - n_R \rightarrow n_G(1) \geq 3-1 \rightarrow n_G \geq 2$