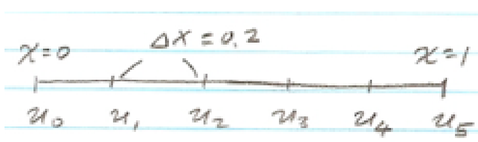
1. (15 pts): (1) 2 pts (2) 5 pts (3) 8 pts

(1) $$\begin{cases} f(x,u) = \frac{du}{dx} = -0.3u^2 + x^2u - 0.5, u(1) = 3, h = 0.2 \\ f(1,3) = -0.3(3)^2 + (1)^2(3) - 0.5 = -0.2 \\ u(1.2) = u(1) + hf\big(1, u(1)\big) = 3 + 0.2(-0.2) = 2.96 \end{cases}$$

(2) $$\begin{cases} u(1.1) = u(1) + (0.1)f\big(1, u(1)\big) = 3 + 0.1(-0.2) = 2.98 \\ f\big(1.1, u(1.1)\big) = -0.3(2.98)^2 + (1.1)^2(2.98) - 0.5 = 0.4417 \\ u(1.2) = u(1) + hf\big(1.1, u(1.1)\big) = 3 + 0.2(0.4417) = 3.0883 \end{cases}$$

(3) $$\left.\begin{matrix} k_1 = f\big(1, u(1)\big) = -0.2, k_2 = f\big(1.1, u(1.1)\big) = 0.4417 \\ k_3 = f\left(1.1, u(1) + \frac{1}{2}k_2(0.2)\right) = f(1.1, 3.0442) = 0.4034 \\ k_4 = f\big(1.2, u(1) + k_3(0.2)\big) = f(1.2, 3.0807) = 1.0890 \end{matrix}\right\} \to \begin{cases} \phi = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 0.4298 \\ u(1.2) = u(1) + h\phi = 3.0860 \end{cases}$$

2.

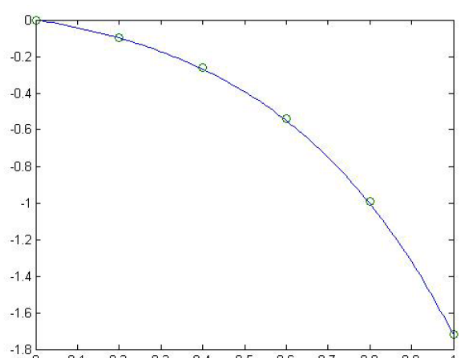


(1) (2+2+2+4 pts)

$$\left.\begin{matrix} \frac{d^2u}{dx^2} \approx \frac{u_{i+1} - 2u_i + u_{i-1}}{(\Delta x)^2} \\ \frac{du}{dx} \approx \frac{u_{i+1} - u_{i-1}}{2(\Delta x)} \end{matrix}\right\} \to \begin{cases} \frac{u_{i+1} - 2u_i + u_{i-1}}{(\Delta x)^2} - 3\frac{u_{i+1} - u_{i-1}}{2(\Delta x)} + 2u_i = 0 \\ \left(\frac{1}{(\Delta x)^2} + \frac{3}{2(\Delta x)}\right)u_{i-1} + \left(2 - \frac{2}{(\Delta x)^2}\right)u_i + \left(\frac{1}{(\Delta x)^2} - \frac{3}{2(\Delta x)}\right)u_{i+1} = 0 \end{cases}$$

$$\left.\begin{matrix} 32.5u_0 - 48u_1 + 17.5u_2 = 0 \\ 32.5u_1 - 48u_2 + 17.5u_3 = 0 \\ 32.5u_2 - 48u_3 + 17.5u_4 = 0 \\ 32.5u_3 - 48u_4 + 17.5u_5 = 0 \end{matrix}\right\} \to \begin{bmatrix} -48 & 17.5 & 0 & 0 \\ 32.5 & -48 & 17.5 & 0 \\ 0 & 32.5 & -48 & 17.5 \\ 0 & 0 & 32.5 & -48 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -17.5(1-e) \end{Bmatrix}$$

(2) (3+2 pts) $$\begin{cases} assume\ u \propto e^{\alpha x} \to \alpha^2 - 3\alpha + 2 = 0 \to \alpha = 1, 2 \to u = Ae^x + Be^{2x} \\ \left.\begin{matrix} u(0) = 0 \to A + B = 0 \\ u(1) = 1 - e \to Ae + Be^2 = 1 - e \end{matrix}\right\} \to \begin{cases} A = e^{-1} \\ B = -e^{-1} \end{cases} \\ u = e^{-1}e^x - e^{-1}e^{2x} = e^{x+1} - e^{2x-1} \end{cases}$$



3. (1) (5 pts)

$$\left.\begin{aligned}\frac{du}{dt} &\approx \frac{u_{i,j+1}-u_{i,j}}{\Delta t}\\ \frac{d^2u}{dx^2} &\approx \frac{u_{i+1,j}-2u_{i,j}+u_{i-1,j}}{(\Delta x)^2}\end{aligned}\right\} \rightarrow \begin{cases}\dfrac{u_{i,j+1}-u_{i,j}}{\Delta t} = 2\dfrac{u_{i+1,j}-2u_{i,j}+u_{i-1,j}}{(\Delta x)^2}\\ u_{i,j+1} = \dfrac{2(\Delta t)}{(\Delta x)^2}u_{i-1,j} + \left(1-\dfrac{4(\Delta t)}{(\Delta x)^2}\right)u_{i,j} + \dfrac{2(\Delta t)}{(\Delta x)^2}u_{i+1,j}\\ P = \dfrac{2(\Delta t)}{(\Delta x)^2}, Q = 1-\dfrac{4(\Delta t)}{(\Delta x)^2}, R = \dfrac{2(\Delta t)}{(\Delta x)^2}\end{cases}$$

(2) $\Delta x = 0.5, \Delta t = 0.1 \rightarrow u_{i,j+1} = 0.8u_{i-1,j} - 0.6u_{i,j} + 0.8u_{i+1,j}$ (2 pts)

$j = 0: u_{5,0} = 1, u_{i,0} = 0$

$$j = 1: \begin{cases}u_{4,1} = 0.8u_{3,0} - 0.6u_{4,0} + 0.8u_{5,0} = 0.8\\ u_{5,1} = 0.8u_{4,0} - 0.6u_{5,0} + 0.8u_{6,0} = -0.6\\ u_{6,1} = 0.8u_{5,0} - 0.6u_{6,0} + 0.8u_{7,0} = 0.8\end{cases}$$ (3 pts)

$$j = 2: \begin{cases}u_{3,2} = 0.8u_{2,1} - 0.6u_{3,1} + 0.8u_{4,1} = 0.8(0.8) = 0.64\\ u_{4,2} = 0.8u_{3,1} - 0.6u_{4,1} + 0.8u_{5,1} = -0.6(0.8) + 0.8(-0.64) = -0.96\\ u_{5,2} = 0.8u_{4,1} - 0.6u_{5,1} + 0.8u_{6,1} = 0.8(0.8) - 0.6(-0.64) + 0.8(0.8) = 1.64\\ u_{6,2} = 0.8u_{5,1} - 0.6u_{6,1} + 0.8u_{7,1} = 0.8(-0.64) - 0.6(0.8) = -0.96\\ u_{7,2} = 0.8u_{6,1} - 0.6u_{7,1} + 0.8u_{8,1} = 0.8(0.8) = 0.64\end{cases}$$ (5 pts)

4. (10 pts)

$$u(x,y) = G(x)H(y) \rightarrow \frac{\partial^2 u}{\partial x \partial y} - 2u = 0 \rightarrow \frac{dG}{dx}\frac{dH}{dy} = 2GH \rightarrow \frac{1}{G}\frac{dG}{dx} = \frac{2H}{\frac{dH}{dy}} = c$$

$$\left.\begin{aligned}\frac{1}{G}dG = cdx \rightarrow lnG = cx + c_1 \rightarrow G = c_1 exp(cx)\\ \frac{1}{H}dH = \frac{2}{c}dy \rightarrow lnH = \frac{2}{c}y + c_2 \rightarrow H = c_3 exp\left(\frac{2}{c}y\right)\end{aligned}\right\} \rightarrow u(x,y) = Cexp\left(cx + \frac{2}{c}y\right)$$

5.

(1) (2 pts * 3 +1)

①Common arithmetic operations (Addition, subtraction, multiplication, division), ②Large computations, ③Adding a large and a small number, ④Subtractive cancellation, ⑤Smearing, ⑥Inner products

(2) (4 pts each)

* Explicit scheme: compute values at each node for a future time based on the present value at the node and its neighbors (approximate the spatial derivative at time level l, conditionally stable, faster than implicit)

* Implicit scheme: approximate the spatial derivative at an advanced time level l+1, unconditionally stable, solve system equations at every time step

6.

(1) $GR_{TM} = \frac{Z_R + Z_S}{Z_S} = \frac{80+40}{40} = 3$ (2 pts)

$J_{vehicle} = m_{body} R_{tire}^2 = 1500 \times 0.3^2 = 135 \text{ kg}\square\text{m}^2 s$ (1 pt)

$J_{eq} = \left(J_{eng} \times GR_{TM}^2 + J_{motor}\right) \times GR_F^2 + J_{vehicle} = \left(0.2 \times 3^2 + 0.1\right) \times 4^2 + 135 = 165.4 \text{kg}\square\text{m}^2$ (2 pts)

(2) $T_{whl} = (T_{eng} \times GR_{TM} + T_{mot}) \times GR_F = \left(50 \times 3 + 80\right) \times 4 = 920 \text{ Nm}$ (2 pts)

$v = 36 \text{ km/h} = \frac{36}{3.6} = 10 \text{ m/s}$, (1 pt) $F_{res} = \frac{1}{2} C_d A_{front} \rho_{air} v^2 + \mu_{roll} m_{body} g = 183 \text{ N}$ (2 pts)

$\alpha_{whl} = \frac{T_{whl} - F_{res} R_{tire}}{J_{eq}} = \frac{920 - 183 \times 0.3}{165.4} = 5.23 \text{ rad/s}^2$, $\therefore a_{veh} = \alpha_{whl} R_{itre} = 1.57 \text{ m/s}^2$ (3 pts)

(3) $v = 36 \text{ km/h} = 10 \text{ m/s}$, $\omega_{mot} = \frac{v}{R_{tire}} GR = \frac{10}{0.3} \times 4 = 133.33 \text{ rad/s}$ (1 pt)

$T_{mot} \omega_{mot} = 80 \times 133.33 = 10,667 \text{ W}$ (1 pt)

$V_{bat} = V_{OCV} - R_{in} I_{bat} = 350 - 0.1 I_{bat}$, $\eta_{motor} V_{bat} I_{bat} = T_{mot} \omega_{mot} \rightarrow 0.09 I_{bat}^2 - 315 I_{bat} + 10667 = 0$

solve $\rightarrow$ $I_{bat} = 34.2 \text{ A}$ (3 pts)

$\therefore SOC = SOC_{ini} - \frac{100}{C_{nom}} \int_0^{5\min} I_{bat}\, dt = 50 - \frac{100}{50,000} \times 34.2 \times 300 = 29.5\%$ (2 pts)

(4) $\omega_{eng} = \frac{v}{R_{tire}} \times GR_{TM} \times GR_F = \frac{10}{0.3} \times 3 \times 4 = 400 \text{ rad/s}$ (1 pts)

At $T_{eng} = 50 \text{ Nm}, \omega_{eng} = 400 \frac{30}{\pi} = 3820 \text{ RPM} \rightarrow \text{BSFC} = 320 \text{ g/kWh}$ (2 pts)

Fuel rate: $\frac{320 \text{ g}}{\text{kWh}} \times \frac{50 \times 400}{1000} \times \frac{1 \text{ h}}{3600 \text{ s}} \times \frac{1 \text{ L}}{860 \text{ g}} = 0.00207 \text{ L/s}$ (2 pts)

Fuel consumption: $\int_0^{5\min} 0.00207 \text{ L/s}\, dt = 0.62 \text{ L}$ (1 pts)

$\therefore \text{FE} = \frac{10 \text{ m/s} \times 300 \text{ s}}{0.62 \text{ L}} \times \frac{1 \text{ m}}{1000 \text{ km}} = 4.84 \text{ km/L}$ (2 pts)

(5) $1 \text{km/L} = \frac{\text{km}}{1.609} \frac{3.785}{\text{L}} = 2.352 \text{ MPG}$ $\therefore \text{FE} = 4.84 \text{ km/L} \times 2.35 \frac{\text{MPG}}{\text{km/L}} = 11.4 \text{ MPG}$ (2 pts)