Midterm Exam

04/26/2011

1. (25 pts) Suppose the 2π-periodic $f(x)$ is a half-length square wave:

$$f(x)=\begin{cases}1 & \text{for } 0<x<\pi/2\\ -1 & \text{for } -\pi/2<x<0\\ 0 & \text{elsewhere in } [-\pi,\pi]\end{cases}$$

(1) Find the Fourier cosine and sine coefficients a_k and b_k of $f(x)$.

(2) Compute $\int_{-\pi}^{\pi}(f(x))^2 dx$ as a number and also as an infinite series using the $a_k^{\,2}$ and $b_k^{\,2}$.

(3) Draw a graph of the integral $I(x)=\int_0^{\pi} f(t)dt$ from –π to π. What are the Fourier coefficients of $I(x)$?

(4) Draw a graph of the integral $D(x)=\frac{df}{dt}$ from –π to π. What are the Fourier coefficients of $D(x)$?

(5) If you convolve $D(x)*I(x)$, check if $D*I=f*f$.

2. (25 pts)

(1) Find the cyclic convolution $\boldsymbol{w}=\boldsymbol{u}*\boldsymbol{v}$ $(N=4)$ of the vectors $\boldsymbol{u}=(1,0,1,0)$ and $\boldsymbol{v}=(0,1,0,1)$.

(2) Find the discrete Fourier coefficients and (4-point DFT) of those vectors c_k and d_k, respectively. From the c_k and d_k find the Fourier coefficients h_k of their convolution $\boldsymbol{w}=\boldsymbol{u}*\boldsymbol{v}$. How does this answer confirm that $\boldsymbol{w}$ in part (1) was correct?

(3) For some 4-component vectors $\boldsymbol{z}$ the cyclic convolution with $\boldsymbol{u}=(1,0,1,0)$ is $\boldsymbol{u}*\boldsymbol{z}=(0,0,0,0)$. Describe the components of $\boldsymbol{z}$ and also its Fourier coefficients (C_0,C_1,C_2,C_3).

3. (25 pts) This question uses the Fourier integral to study

$$-\frac{d^2u}{dx^2}+u(x)=\begin{cases}1 & \text{for } -1\le x\le 1\\ 0 & \text{for } |x|\le 1\end{cases}$$

(1) Take Fourier transforms of both sides to find a formula for $\hat{u}(k)$.

(2) What is the decay rate of this $\hat{u}$? At what points x is the solution $u(x)$ not totally smooth? Describe $u(x)$ at those points: delta, jump in $u(x)$, jump in du/dx, jump in d^2u/dx^2, or what?

(3) We know that the Green's function for this equation (when the right side is $\delta(x)$) is

$$G(x)=\frac{1}{2}e^{-|x|}=\begin{cases}\frac{1}{2}e^{-x} & \text{for } x\ge 0\\ \frac{1}{2}e^{x} & \text{for } x\le 0\end{cases}$$

Find the solution $u(x)$ at the particular point $x=2$.

4. (25 pts) Describe the first order and the second order methods to solve $u'=-10u$, respectively, including stability limits.