Final Exam

06/12/2013

1. (30 pts) Consider the following initial value problem and answer the following:

$$u_t = u_x, \quad 0 < x < 2\pi, \quad t > 0$$
$$u(x,0) = \sin(x)$$
$$u(0,t) = u(2\pi,t)$$

(1) exact solution to this problem

(2) v_j^{n+1} for three numerical methods (notations: $v_j^n = u(x_j, t_n)$, $k = \Delta t$, $h = \Delta x$, $\lambda = k/h$)

a. Leap Frog b. Upwind c. Lax-Wendroff

(3) stable condition for all three schemes

(4) order of accuracy of each method (refer to Table 1)

Table 1: Relative error (in %) with respect to the exact solution

k	lf	uw	lw
0.050265482	6.468230633	7.207733750	6.434407102
0.025132741	3.194120693	3.661412223	3.183507851
0.012566371	1.585770992	1.844733852	1.582493879
0.006283185	0.789699788	0.925650435	0.788705573
0.003141593	0.393935880	0.463552334	0.393648608
0.001570796	0.196713247	0.231922525	0.196626404

2. (20 pts) Consider the heat equation or diffusion equation (parabolic PDE) $u_t = u_{xx}$ and compare the following numerical methods: (1) explicit method (2) fully implicit method (3) Crank-Nicolson implicit method

3. (25 pts) Consider solving large systems:

(1) List direct elimination methods and describe their characteristics.

(2) List iteration methods and describe their characteristics.

4. (25 pts) The shortest curve connecting two points is a straight line. Suppose we cannot go in a straight line because of a constraint. When the constraint is $\int u(x)dx = A$, find the shortest curve $u(x)$ between $u(0) = a$ and $u(1) = b$ that has area A below it.