Midterm Exam

04/15/2013

1. (25 pts) $f(x)$ is the 2π-periodic function that equals $e^{ix/2}$ from $x=0$ to $x=2\pi$.

(1) What is the value of $f(x)$ at $x=3\pi$? Is that the same as $e^{3\pi i/2}$?

(2) Find the Fourier coefficients c_k in $f(x)=\sum_{-\infty}^{\infty} c_k e^{ikx}$.

(3) What is the decay rate of those coefficients c_k? How is that decay rate explained using the smoothness properties of $f(x)$?

2. (25 pts)

(1) Set up 4 equations for the numbers $a_0,\ a_1,\ a_2,\ a_3$ so that the cubic $p(x)$ has the values $b_0,\ b_1,\ b_2,\ b_3$ at the 4 points $x=1,\ i,-1,-i$:

$$p(x)=a_0+a_1x+a_2x^2+a_3x^3=\begin{cases}b_0\\ b_1\\ b_2\\ b_3\end{cases} \text{ at } x=\begin{cases}1\\ i\\ -1\\ -i\end{cases}$$

(2) Those 4 equations could be written in matrix-vector form as $\boldsymbol{Ma}=\boldsymbol{b}$. What is the inverse of that matrix $\boldsymbol{M}$?

(3) Conclusion: Matching these values $b_0,\ b_1,\ b_2,\ b_3$ at these points is the same as

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3. (25 pts)

(1) Find the discrete Fourier coefficients $c_0,\ c_1,\ c_2$ of the discrete values $f_0=2,\ f_1=-1,\ f_2=-1$.

(2) What is the cyclic convolution $c\otimes c$ of $c=(c_0,\ c_1,\ c_2)$ with itself?

(3) Check the convolution rule by multiplying component by component to get $f.*f=(f_0^2, f_1^2, f_2^2)$ and finding the discrete coefficients of this vector. The result should be (and is) the same as

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4. (25 pts) A Helmholtz equation (with c = real constant) looks like

$$-\frac{d^2u}{dx^2}+u(x)-c^2u(x)=f(x) \quad \text{for} \quad -\infty<x<\infty$$

(1) With Fourier Integrals, transform this into an equation for $\hat{u}(k)$. Find $\hat{u}(k)$ in terms of $\hat{f}(k)$ and $u(x)$.

(2) Which values of c lead to difficulty in working with your formula for $\hat{u}(k)$? What is the problem?

(3) With $c=1$ and $f(x)=\delta(x)$, show directly that $-u''(x)=\delta(x)$ has no solution with $u(x)\to 0$ as $x\to\infty$.