Midterm Exam

04/25/2016

1. (25 pts) Consider the following boundary-value problem:

$$\begin{cases} \text{Differential equation:} & -(\alpha u')' + \gamma u = f \ \text{ in } \ \Omega = (a,b) \\ \text{Neumann boundary condition:} & \alpha u' n_x = \hat{q} \ \text{ in } \ \Gamma_q = \{a\} \\ \text{Dirichlet boundary condition:} & u = \hat{u} \ \text{ on } \ \Gamma_u = \{b\} \end{cases}$$

$\Omega = (a,b)$ be an open set (an open interval in 1D problems)

Γ be the boundary of Ω, that is, $\Gamma = \{a,b\}$

$\Gamma = \Gamma_q \cup \Gamma_u$ where $\Gamma_q = \{a\}$ and $\Gamma_u = \{b\}$ are disjoint parts of the boundary $(\Gamma_q \cap \Gamma_u = \varnothing)$

(1) Write the weighted-residual statement for the domain equation.

(2) Trade differentiation from u to the weighting function using integration by parts.

(3) Derive weak form using the Neumann boundary condition. Explain why the Neumann boundary condition is also called the natural boundary condition.

2. (25 pts)

(1) The 2π-periodic function $F(x)$ equals 1 for $0 \le x < \pi$ and equals 0 for $\pi \le x < 2\pi$. Find its Fourier coefficients c_k using complex exponentials: $F(x) = \sum_{-\infty}^{\infty} c_k e^{ikx}$

Write out the terms for $k = -1,\ 0,\ 1$. What is the decay rate of the c_k as $k \to \infty$? How do you see this from the function $F(x)$?

(2) The energy equality connects $\int |F(x)|^2\, dx$ with $\sum |c_k|^2$. What is this equation for our particular $F(x)$? Find a formula for π.

(3) What is the derivative of this $F(x)$? Draw the graph of dF/dx. What is the complex Fourier series for dF/dx? What is the decay rate of the coefficients? WHY?

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3. (25 pts)

(1) These matrix-vector multiplications Cx and Cy are the cyclic convolution of which vectors?

$$Cx = \begin{bmatrix} 5 & -1 & -1 & -1 \\ -1 & 5 & -1 & -1 \\ -1 & -1 & 5 & -1 \\ -1 & -1 & -1 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \end{bmatrix} \text{ and } Cy = \begin{bmatrix} 5 & -1 & -1 & -1 \\ -1 & 5 & -1 & -1 \\ -1 & -1 & 5 & -1 \\ -1 & -1 & -1 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 6 \\ -6 \\ 6 \\ -6 \end{bmatrix}$$

Take the Discrete Fourier Transform of all three vectors c, x, y. Call those transforms $\hat{c}, \hat{x}, \hat{y}$.

(2) Convert those two cyclic convolutions Cx and Cy into component-by-component multiplications of the transforms. The answer uses numbers.

(3) Apparently this $y = (1, -1, 1, -1)$ is an eigenvector with $\lambda = 6$. Multiply any circulant matrix C times y to find the eigenvalue:

$$Cy = \begin{bmatrix} c_0 & c_3 & c_2 & c_1 \\ c_1 & c_0 & c_3 & c_2 \\ c_2 & c_1 & c_0 & c_3 \\ c_3 & c_2 & c_1 & c_0 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} = \lambda y$$

How is λ connected to the transform $\hat{c}$ of $c = (c_0, c_1, c_2, c_3)$?

4. (25 pts) Suppose I want to use the Fourier Integral transform to solve this fourth-order differential equation:

$$\frac{d^4 u}{dx^4} = \delta(x) - 4\delta(x-1) + 6\delta(x-2) - 4\delta(x-3) + \delta(x-4)$$

(1) Take the transform of all terms to find $\hat{u}(k)$.

(2) The solution $u(x)$ will have jumps in which derivative (?) at which points $x(?)$.

(3) What is $u(x)$ between $x = 0$ and $x = 1$? Do you recognize $u(x)$?